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Cohomological invariants of SK_1

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Promotoren:
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Prof. Joost van Hamel
Prof. Willem Veys

Proefschrift voorgedragen
tot het behalen van de
graad van Doctor in de
Wetenschappen (Wiskunde)

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Avec tout mon respect et ma considération pour
la communauté mathématique française et russe.

С глубоким уважением и почитанием к
французскому и русскому математическому
сообществу.

Dankwoord

"The more you know,
the more you realise
how little you know."
— Daodejing

Negen jaar intensief wiskunde studeren verandert een mens. Het proces gaat geleidelijk aan, maar je merkt de effecten van het abstract redeneren in je dagelijkse leven. Je begint zowat alles dieper te analyseren. Na negen jaar wiskunde besef ik dat ik zeker niet nog alles in de wiskunde gezien heb, maar ik wil graag met de opgedane ervaringen andere horizonten gaan verkennen. Dit is dan ook de uitgelezen kans om enkele mensen te danken die me de gesteund hebben in mijn trektocht doorheen de wiskunde.

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Tim Wouters
Mei 2010

Abstract

The subject of this thesis is the group functor $\mathbf{SK}_1(A)$ for a central simple algebra A over a field k. We concentrate on cohomological invariants of $\mathbf{SK}_1(A)$ which can - as claimed by Suslin [Sus] - help to explain Platonov's examples of non-trivial SK_1 [Pla]. Most of the existing ones restrict to central simple algebras A of $\mathrm{ind}_k(A) \in k^\times$.

A first aim of this thesis is to generalise these invariants to any central simple algebra (so that we can drop the condition on the index). For that purpose, we use a lift from positive characteristic to characteristic zero. As the condition on the index is empty in characteristic zero, we can use the existence of the invariants in characteristic zero and then specialise in a proper way to positive characteristic. This involves notions of p-rings, Kato's logarithmic differentials, and some important results by Kahn and Merkurjev.

We also compare this construction with an invariant of $\mathbf{SK}_1$ for biquaternion algebras as defined by Knus-Merkurjev-Rost-Tignol [KMRT, §17]. This invariant also does not have the condition on the index. For biquaternion algebras in characteristic 2, we prove this invariant essentially equals a generalised invariant of Suslin. We finish this thesis by proving the non-triviality of an invariant of $\mathbf{SK}_1(A)$ recently introduced by Kahn [Kah3]. We also give a formula for the value on the centre of the tensor product of two symbol algebra, which generalises a formula from Merkurjev for the centre of two biquaternions [Mer2].

In an appendix we describe the behaviour of the so-called elementary obstruction under the Weil restriction. The elementary obstruction can determine whether a variety contains no rational points. In this appendix we prove the invariance of this elementary obstruction under taking a Weil restriction of scalars. This was the content of a first paper of the author. Although the subject is quite different from the core of this thesis, the methods used are very similar.

Samenvatting

In deze doctoraatsverhandeling bestuderen we de groepsfunctor $\mathbf{SK}_1(A)$ voor een centrale enkelvoudige algebra A. Daarbij concentreren we ons op cohomologische invarianten van deze groepsfunctor. Zoals verondersteld door Suslin [Sus], is de hoop dat deze (onder meer) Platonovs voorbeelden van niet-triviale SK_1 kunnen verklaren. Het merendeel van de reeds bestaande invarianten beperkt zich steeds tot centrale enkelvoudige algebra's A met $\mathrm{ind}_k(A) \in k^\times$.

In deze thesis introduceren we een methode om deze invarianten te veralgemenen (zodat we de voorwaarde op de index kunnen laten vallen). Hiervoor gebruiken we een opheffing van positieve karakteristiek naar karakteristiek nul. Aangezien de voorwaarde in karakteristiek nul niet-bestaande is, kunnen we het bestaan van invarianten in karakteristiek nul gebruiken om via een specialisatie invarianten in positieve karakteristiek te verkrijgen. Dit vereist het gebruik van p-ringen, logaritmische differentialen (op zijn Kato's) en belangrijke hulpresultaten van Kahn en Merkurjev.

We vergelijken deze constructie ook met een invariant van $\mathbf{SK}_1$ voor biquaternionen ingevoerd door Knus-Merkurjev-Rost-Tignol [KMRT, §17]. Deze invariant heeft ook geen voorwaarde op de index. We bewijzen dat deze gelijk is aan de nieuw geconstrueerde invariant. Tot slot tonen we aan dat een specifieke invariant van Kahn niet triviaal is voor het product van twee symboolalgebra's op zijn Platonovs. Tevens veralgemenen we een formule van Merkurjev voor de waarde op het centrum van biquaternionen [Mer2] naar het het tensorproduct van twee symboolalgebra's.

In een appendix beschrijven we het gedrag van de elementaire obstructie van een variëteit onder de weilrestrictie. De elementaire obstructie kan bepalen dat een variëteit geen rationale punten heeft. We bewijzen dat de elementaire obstructie invariant is onder het nemen van de weilrestrictie. Dit was de inhoud van een eerste artikel van de auteur. Alhoewel het onderwerp op zich verschillend is van de rest van de thesis, zijn de gebruikte methoden gelijkaardig.

Contents

Dankwoord **iii**

Abstract **v**

Samenvatting **vi**

Contents **vii**

Notations and conventions **ix**

Introduction **1**

I.1 SK_1 of a central simple algebra 4

I.2 Wang's theorem and Suslin's conjecture 7

I.3 Reductions of the problem 9

I.4 Overview of the thesis 11

1 Cycle modules and invariants **13**

1.1 Cohomology groups 14

1.2 Cycle modules 18

1.3 Invariants à la Merkurjev 23

1.4 Invariants of $\mathbf{SK}_1$ 24

2 Lifting and specialising invariants **29**

2.1 Moderate case 29

2.2 Wild case 40
2.3 General case 56
2.4 Some remarks 58

3 Comparing invariants **63**
3.1 Invariants for biquaternion algebras 63
3.2 Kahn's invariant 76

Conclusion **89**

A Verification of cycle module rules **93**

B Elementary obstruction and Weil restriction **101**
B.1 Introduction 101
B.2 Product varieties 104
B.3 Weil restriction 109

Bibliography **115**

Glossary **121**

Index **125**

Notations and conventions

Throughout this thesis we use some standard notations and conventions of the field of research (unless explicitly otherwise stated). The reader can come back to these pages when he wants to recall them. We also refer to the glossary for a comprehensive list of the notations in use.

- For a field k, we denote by $\overline{k}$ an algebraic closure and by $k_s \subset \overline{k}$ a separable closure. Furthermore, $\Gamma_k = \mathrm{Gal}(k_s/k)$ is the absolute Galois group, $\mathrm{cd}(k)$ (resp. $\mathrm{cd}_p(k)$) is the (p-)cohomological dimension (for p a prime), $k((t_1))\ldots((t_n))$ is the n-fold iterated Laurent series field over k in variables $t_1, \ldots, t_n$, and $\mathbb{G}_m$ is the multiplicative group $\mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$.

- We use standard notations for the following categories: the category $\mathfrak{Sets}$ of sets, the category k-$\mathfrak{fields}$ of field extensions of a field k, the category $\mathfrak{Groups}$ of groups, and the category $\mathfrak{Ab}$ of abelian groups.

- We always suppose k-algebras to be associative, to have a multiplicative identity 1, and to be finite dimensional over k.

- If A is a k-algebra and if K is a field extension of k, we denote by A_K the K-algebra $A \otimes_k K$ obtained from A by base extension to K. Likewise, if X is a k-scheme, X_K is the K-scheme $X \times_k K (= X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K))$ obtained from X by base extension to K. Furthermore, $X(K)$ is the set of K-rational points of X.

- A prime factorisation $p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ of a (positive) integer m is always supposed to be primitive (i.e. $m = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$, with p_i primes, $e_i \geq 1$ integers for $1 \leq i \leq n$, and $p_i \neq p_j$ for any $1 \leq i < j \leq r$).

- For an integer $m > 0$ invertible[1] in a field k, we denote by μ_m the Γ_k-module of m-th roots of unity in k_s. If one forgets about the Γ_k-action, μ_m is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Unless k contains a primitive m-th

[1] We use this expression for brevity; it actually comes down to requiring $\gcd(m, p) = 1$ if $\mathrm{char}(k) = p > 0$ and $m > 0$ arbitrary if $\mathrm{char}(k) = 0$.

root of unity (so in particular if $m = 1, 2$), the isomorphism does not continue to an isomorphism of Γ_k-modules (if $\mathbb{Z}/m\mathbb{Z}$ is equipped with the trivial Γ_k-action). We write $\mu_m(k)$ for the m-th roots of unity inside k itself (so that it can be viewed as the rational points of the appropriate sheaf). We also use the latter notation for arbitrary rings.

- The cohomology groups used are Galois (or étale) cohomology groups.

- A discrete valuation v on a field F is supposed to be non-trivial and of rank 1. We denote the valuation ring by $\mathcal{O}_v$ and the residue field by $\kappa(v)$. The maximal unramified extension of F with respect to v is denoted as F_{nr}. If $x \in \mathcal{O}_v$, its residue in $\kappa(v)$ is $\bar{x}$. This notation is also used for other objects with natural residues (induced by a discrete valuation on a field). We also distinguish two different cases of discrete valuation fields depending on the characteristics: the *equicharacteristic case* if $\mathrm{char}(F) = \mathrm{char}(\kappa(v))$ and the *mixed characteristic case* if $\mathrm{char}(F) = 0$ and $\mathrm{char}(\kappa(v)) = p$.

- For any group G and integer m, we denote by ${}_mG$ the m-torsion points of G.

- For any scheme X of finite dimension and integer $i \geq 0$, we denote by $X^{(i)}$ the points of codimension i of X. An algebraic k-group is a smooth affine group scheme over k of finite type.

As for references, the author tries to include the exact reference to the theorem in use, unless the cited article lacks numbered theorems. In the latter case, no further details probably means the article has one main theorem, which is the one referred to.

Introduction

"Une conjecture est d'autant plus utile qu'elle est plus précise, et de ce fait testable sur des exemples."
— Jean-Pierre Serre

In this thesis we are interested in *central simple algebras* over a field k. These k-algebras have centre equal to $k = (k.1)$ (*central*) and have no two-sided ideals except for the trivial ones, 0 and the algebra itself (*simple*). Unless otherwise stated, in this introduction we always consider A to be a central simple k-algebra.

Very important examples of central simple algebras are *central division algebras*; these are central k-algebras containing a multiplicative inverse for all of its non-zero elements. More generally, every matrix algebra $M_n(D)$ over a central division algebra is a central simple algebra. The following alternative definition shows that these are actually all examples of central simple algebras.

Theorem I.1 (see e.g. [GS, §§2.1 - 2.2])

Let A be an algebra over a field k, then the following conditions are equivalent:

(i) A is a central simple k-algebra,

(ii) there exists a central division algebra D over k such that $A \cong M_r(D)$ as k-algebras (r some integer),

(iii) there exists a field extension K/k such that $A_K \cong M_n(K)$ as K-algebras (n some integer).

Remark I.2 – The equivalence (i)↔(ii) is commonly known as *Wedderburn's theorem* as it was proved by Wedderburn in 1908 [Wed]. Even more, the central division algebra is uniquely determined up to isomorphism.

Wedderburn's theorem is used to prove the equivalence (i)$\leftrightarrow$(iii). A field K satisfying condition (iii) is called a *splitting field* of A. It can be proved that $\overline{k}$, k_s, and even a finite extension of k suffice. The choice of this finite splitting field depends (of course) heavily on A (and not just on k).

This theorem gives rise to the definition of the *Brauer group* $\mathrm{Br}(k)$ of a field k. Two central simple k-algebras A and B are said to be *Brauer-equivalent* ($A \sim_{\mathrm{Br}} B$) if there exist two positive integers n, m such that $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$ as k-algebras. We denote the equivalence class of A by $[A]$, the *Brauer class* of A. For two central simple k-algebras A and B, the tensor product $A \otimes_k B$ is again a central simple k-algebra by Theorem I.1 (iii). It can be proved that this endows $\mathrm{Br}(k)$ with the structure of an abelian group. The identity element is the class of k (or $M_n(k)$). The inverse of A is the opposite algebra

$$A^{\mathrm{op}} = \{a^{\mathrm{op}} \in A \,|\, a \in A\}$$

with addition and (scalar) multiplication defined by

$$a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}}, \quad \lambda a^{\mathrm{op}} = (\lambda a)^{\mathrm{op}}, \text{ and } a^{\mathrm{op}} \cdot b^{\mathrm{op}} = (b \cdot a)^{\mathrm{op}},$$

for $a, b \in A$ and $\lambda \in k$. See [GS, Prop. 2.4.8] for a proof. By Theorem I.1 (ii), every Brauer class contains a central division algebra unique up to isomorphism. Another very well known description of the Brauer group is by Galois cohomology: $\mathrm{Br}(k) \cong H^2(k, k_s^\times)$ (ibid., §4.4).

For a field extension K of k, there exists a morphism $\mathrm{Br}(k) \to \mathrm{Br}(K)$ sending the class $[A]$ to the class $[A_K]$. Note that because of Theorem I.1 (iii), it is clear that the base extension of a central simple algebra is still a central simple algebra. By $\mathrm{Br}(K/k)$ we denote $\ker(\mathrm{Br}(k) \to \mathrm{Br}(K))$, i.e. the subgroup of $\mathrm{Br}(k)$ consisting of the classes of central simple algebras which split after base extension to K. So e.g. $\mathrm{Br}(k_s/k) = \mathrm{Br}(k)$. For more facts and trivia about central simple algebras we refer to some standard works as [Dra, Ch. 1 & 2], [GS, Ch. 2 & 4], [KMRT, §1], and others.

In particular, all of this gives rise to the definition of three integers attached to a central simple algebra.

Definition I.3

Let A be a central simple algebra over a field k. Define the following integers:

- the *degree of* A as $\deg(A) = \sqrt{\dim_k(A)}$,
- the *period of* A as the order $\mathrm{per}_k(A)$ of $[A]$ in $\mathrm{Br}(k)$, and
- the *index of* A as $\mathrm{ind}_k(A) = \sqrt{\dim_k(D)}$, where D is the unique central division k-algebra Brauer-equivalent to A.

Remark I.4 – The fact that $\dim_k(A)$ is a square follows by Theorem I.1 (iii) since $\dim_K(A_K) = \dim_k(A)$ for any field extension K of k. The fact that the order of $[A] \in \mathrm{Br}(k)$ is finite, follows by the isomorphism $\mathrm{Br}(k) = H^2(k, k_s^\times)$ and calculations with Galois cohomology using restrictions and corestrictions (see e.g. [GS, §4.4]). In the notation for period and index, we deliberately used a subscript for the base field as it is not invariant under base extension. The degree however is fixed under extensions of the base field.

It can also be proved that $\mathrm{per}_k(A)$ divides $\mathrm{ind}_k(A)$ and that they have the same prime factors (ibid., Prop. 4.5.13). A whole field of study is dedicated to determining the possible values of $\mathrm{ind}_k(A)/\mathrm{per}_k(A)$. This problem is commonly known as the *period-index problem*. For sure, the index and period are not always equal (see e.g. Example I.10). See (ibid., Rem. 4.5.5) for some comments on this problem. We do not go into details on this subject, we rather study other constructions related to central simple algebras.

Example I.5 – Let us first give some important examples of central simple algebras.

(i) *Cyclic algebras*
Suppose K is a cyclic field extension of k of degree n (i.e. $\mathrm{Gal}(K/k) \cong \mathbb{Z}/n\mathbb{Z}$). Let σ be any generator of $\mathrm{Gal}(K/k)$ and $a \in k^\times$. We define the cyclic algebra $(K/k, \sigma, a)$ as the k-algebra generated by K and a variable x satisfying the relations $x^n = a$ and $xc = \sigma(c)x$ for any $c \in K$. So we can write this cyclic algebra as $\oplus_{i=0}^{n-1} Kx^i$ with multiplication defined as above. Also $\deg(K/k, \sigma, a) = n$ and K is a splitting field of $(K/k, \sigma, a)$ (see [GS, §2.5] where also another description of cyclic algebras is given).

(ii) *Symbol algebras*
Let $n \in k^\times$ be an integer and suppose k contains an n-th primitive root of unity ξ_n. For any $a, b \in k^\times$, we define the *symbol algebra* $(a, b)_n$ as the central simple k-algebra generated by variables x and y satisfying $x^n = a, y^n = b$, and $xy = \xi_n yx$. Clearly $\deg (a, b)_n = n$. Note that this algebra depends on the choice of the primitive root of unity [Dra, §11, Lem. 6].[2]

(iii) *p-algebras*
If k is a field of $\mathrm{char}(k) = p > 0$, then for $a \in k$ and $b \in k^\times$ we define the *p-algebra* $[a, b)_p$ as the central simple k-algebra generated by u and v satisfying $u^p - u = a, v^p = b$, and $uv = v(u + 1)$. Also $\deg [a, b)_p = p$. These p-algebras play the role of symbol algebras with degree equal to $\mathrm{char}(k) = p > 0$ as in this case k lacks (non-trivial) primitive roots of unity.

Both symbol division algebras and division p-algebras are a special case of cyclic algebras [GS, Cor. 2.5.5 & Rem. 2.5.6]. If k contains an n-th primitive root of unity and if $K = k(\sqrt[n]{a})$ for $a \in k^\times$, then any symbol division algebra $(a, b)_n$ is k-isomorphic to $(K/k, \sigma, b)$ for a well chosen σ. In case $n = p = \mathrm{char}(k)$ and if K is the cyclic Galois extension defined by $x^p - x - a$, then any division p-algebra $[a, b)_p$ is k-isomorphic to $(K/k, \sigma, b)$ for a well chosen σ.

Algebras of the form $(a, b)_2$ or $[a, b)_2$ are called quaternion algebras. The name comes from the fact that Hamiltonian quaternions are retrieved for $k = \mathbb{R}$ and $a, b = -1$. As usual for quaternion algebras, we drop the subscript 2. If we want to treat both symbol and p-algebras, we loosely speak about algebras of the form $[(a, b)_p$ as Draxl does in [Dra, §14]. We trust on the reader's good-will to make the proper assumptions on a, b and the characteristic of the base field k.

I.1 SK_1 of a central simple algebra

Our interest in this thesis goes to the functor $\mathbf{SK}_1(A)$. To define it, we need the notion of the reduced norm of A. We recall the notions without giving (rigorous) proofs, see e.g. [Dra, §22] and [GS, §§2.6 & 2.8] for details.

[2]One could incorporate the chosen root of unity in the notation. In this text we do not explicitly work with symbol algebras defined with different primitive roots of unity. Hence we use this more elementary notation which actually does not show the true colours of the algebra.

Definition I.6

Let A be a central simple k-algebra. A splitting field K of A defines a multiplicative map, called the *reduced norm* $Nrd_{A/k}$ as composition of

$$A \overset{\mathrm{id}\otimes 1}{\hookrightarrow} A \otimes_k K \cong M_n(K) \overset{\det}{\to} K,$$

which can be proved to be independent of the splitting field and to have values in k. Even more, the elements in A with reduced norm in $k^\times$ are exactly the units of A.

Using a splitting field K of A, the embedding $id \otimes 1 : A \to A \otimes_k K$, and the corresponding terms for matrices, one can also define a *reduced trace* $\mathrm{Trd}_{A/k} : A \to k$ and a *reduced characteristic polynomial* $\mathrm{Prd}_{a/k}(X) \in k[X]$ of an element $a \in A$. Even more, for any $a \in A$ the reduced norm $\mathrm{Nrd}_{A/k}(a)$ and trace $\mathrm{Trd}_{A/k}(a)$ can be expressed as coefficients of $\mathrm{Prd}_{a/k}(X)$:

$$\mathrm{Prd}_{a/k}(X) = X^n - \mathrm{Trd}_{A/k}(a)X^{n-1} + b_{n-2}X^{n-2} + \ldots + b_1 X + (-1)^n \mathrm{Nrd}_{A/k}(a). \tag{I.1}$$

This is a generalisation of the expression of the norm $N_{K/k}(x)$ and trace $\mathrm{Tr}_{K/k}(x)$ of an element x of a finite extension K of k as coefficients of its minimal polynomial [Lan, Ch. VI, Thm. 5.1].

The original construction of $\mathrm{SK}_1(A)$ uses $K_1(A)$, the *first K-group* of A or *Whitehead group* of A. Let R be any ring, then we can consider the tower of embeddings

$$\mathrm{GL}_1(R) \subset \mathrm{GL}_2(R) \subset \ldots \subset \mathrm{GL}_n(R) \subset \mathrm{GL}_{n+1}(R) \subset \ldots,$$

where the injections are given by identifying any $A \in \mathrm{GL}_n(R)$ with the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R).$$

Then define

$$\mathrm{GL}_\infty(R) = \bigcup_{n>0} \mathrm{GL}_n(R) \quad \text{and} \quad K_1(R) = \mathrm{GL}_\infty(R)/[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)].$$

For any positive integer n, there is an isomorphism $K_1(R) \cong K_1(M_n(R))$, called the *Morita isomorphism* . This isomorphism is induced by the map

$$M_m(R) \to M_{nm}(R) : A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{nm-m} \end{pmatrix},$$

where m is any positive integer. So using Wedderburn's theorem, we see that for our central simple k-algebra A, the isomorphism class of $K_1(A)$ only depends on the Brauer class of A.

Furthermore, it is also possible to define a reduced norm map $\mathrm{Nrd}_{K_1(A)}$: $K_1(A) \to k^\times$ using the composition

$$\mathrm{GL}_n(A) \cong \mathrm{GL}_1(M_n(A)) \xrightarrow{\mathrm{Nrd}_{M_n(A)}} k^\times.$$

This brings us to the definition of $\mathrm{SK}_1(A)$.

Definition I.7

For any central simple k-algebra A, the *reduced Whitehead group* is

$$\mathrm{SK}_1(A) = \ker(\mathrm{Nrd}_{K_1(A)}).$$

Suppose that D is the unique central division algebra Brauer-equivalent to A (so $A \cong M_n(D)$ for an integer n). Then note that the isomorphism $K_1(A) \cong K_1(D)$ from above also leads to an isomorphism $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(D)$, what we call the *Morita invariance of SK_1* (i.e. $\mathrm{SK}_1(A)$ only depends on the Brauer class of A). Also by definition, the composition

$$A^\times \to K_1(A) \xrightarrow{\mathrm{Nrd}_{K_1(A)}} k^\times$$

coincides with the reduced norm map $A^\times \to k^\times$. Denote

$$\mathrm{SL}_1(A) = \{a \in A \,|\, \mathrm{Nrd}_{A/k}(a) = 1\},$$

the *special linear group of A* . If $A = M_n(k)$, then $\mathrm{SL}_1(A)$ coincides with $\mathrm{SL}_n(k)$. We clearly have an injection

$$\mathrm{SL}_1(A)/[A^\times, A^\times] \hookrightarrow \mathrm{SK}_1(A)$$

which is known to be bijective for central division algebras. The morphism

$$\mathrm{SL}_1(D) \to \mathrm{SL}_1(A) : B \to \begin{pmatrix} B & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{SL}_1(D)/[D^\times, D^\times] \arrow[r, "\cong"] \arrow[d] & \mathrm{SK}_1(D) \arrow[d, "\cong"] \\
\mathrm{SL}_1(A)/[A^\times, A^\times] \arrow[r, hook] & \mathrm{SK}_1(A),
\end{tikzcd}$$

giving us the following property.

Proposition I.8

For any central simple k-algebra A, there is an isomorphism

$$\mathrm{SK}_1(A) \cong \mathrm{SL}_1(A)/[A^\times, A^\times].$$

Remark I.9 – Since $\mathrm{Nrd}_{A/k}$ is multiplicative, it is straightforward to see that the commutators of $A^\times$ are part of $\mathrm{SL}_1(A)$ so that this quotient does make sense.

In the following, we use this description when we speak about $\mathrm{SK}_1(A)$.

I.2 Wang's theorem and Suslin's conjecture

In 1943, Tannaka and Artin independently asked whether $\mathrm{SK}_1(A)$ is always trivial or not, i.e. whether any element of $\mathrm{SL}_1(A)$ is always a commutator in $A^\times$ or not [NM, Wan]. In 1950, Wang proved the triviality of $\mathrm{SK}_1(A)$ if $\mathrm{ind}_k(A)$ is square-free [Wan]. During more than 30 years, one tried to solve the *Tannaka-Artin problem* by proving the triviality of SK_1 in full generality.

Fortunately for the sake of interest of this thesis, in 1976 Platonov came up with examples of non-trivial SK_1 using valuation theory [Pla]. Let us recall quickly the most important of his examples.

Example I.10 (ibid., Thms 4.7 & 5.9) – Let k be local field (e.g. $\mathbb{F}_p((x))$ or $\mathbb{Q}_p$ for a prime p) and let K_1, K_2 be two cyclic extensions of degree n over k which are linearly disjoint and set $K = K_1 \otimes_k K_2 = K_1 \cdot K_2$ (as of [Bou, A

V.13]). Let σ_1 (resp. σ_2) be a generator of $\mathrm{Gal}(K_1/k)$ (resp. $\mathrm{Gal}(K_2/k)$). Now let $F = k((t_1))((t_2))$, $F_1 = K_1((t_1))((t_2))$, and $F_2 = K_2((t_1))((t_2))$. Then Platonov proves that

$$A = (F_1/F, \sigma_1, t_1) \otimes_F (F_2/F, \sigma_2, t_2)$$

is a division F-algebra and $\mathrm{SK}_1(A) \cong \mathbb{Z}/n$. To prove the latter, he uses an isomorphism

$$\mathrm{SK}_1(A) \cong \mathrm{Br}(K/k)/(\mathrm{Br}(K_1/k)\mathrm{Br}(K_2/k)). \tag{I.2}$$

Platonov also gives central simple k-algebras A with $\mathrm{SK}_1(A) = 0$, but $\mathrm{SK}_1(A_K) \neq 0$ where K is a particular field extension of k (ibid. Corr. 6.3). Furthermore, he also proves that for any positive integers i, p one can find fields k and central simple k-algebras A such that $\mathrm{SK}_1(A) \cong (\mathbb{Z}/p\mathbb{Z})^i$ (ibid. Thm. 6.2). The first encounter of these situations was striking.

These examples inspired Suslin to refine the Tannaka-Artin problem to a conjecture he stated in 1991. For this conjecture, he rather uses a functorial version of SK_1.

Definition I.11

For a field k and a central simple k-algebra A, define

$$\mathbf{SK}_1(A) : k\text{-}\mathfrak{fields} \to \mathfrak{Ab} : K \mapsto \mathbf{SK}_1(A)(K) = \mathrm{SK}_1(A_K).$$

Conjecture I.12 (Suslin [Sus, Intro.])

Let A be a central simple k-algebra, then $\mathbf{SK}_1(A) = 0$ if and only if $\mathrm{ind}_k(A)$ is square-free.

Remark I.13 – By $\mathbf{SK}_1(A) = 0$, we mean of course that $\mathbf{SK}_1(A)(K) = 0$ for any field extension K of k. By Wang's theorem it is turned into a necessity statement as $\mathrm{ind}(A_K) \mid \mathrm{ind}(A)$ for any field extension K [Pie, Prop. 13.4]. Furthermore, by Wang's theorem it also follows that $\mathbf{SK}_1(A)(K) = 0$ if K is a splitting field of k. Also if K is a finite field extension of k of degree prime to $\mathrm{ind}_k(A)$, then $\mathbf{SK}_1(A)(k) \to \mathbf{SK}_1(A)(K)$ is an injection [Dra, §23, Lem. 3].

Due to Proposition I.8, this problem is related to the linear algebraic k-group

$$\mathbf{SL}_1(A) = \mathrm{Spec}\Big(k[X_1, \ldots, X_{n^2}]/I\Big),$$

where $X_1, \ldots, X_{n^2}$ are variables parametrising the coefficients of the elements of A with respect to a k-vector space basis and I is the ideal generated by the polynomial in the X_i defined by requiring that the reduced norm equals 1. Of course $\mathbf{SL}_1(A)(K) = \mathrm{SL}_1(A \otimes_k K)$.

Suslin's conjecture translates into a conjecture whether or not $\mathrm{ind}_k(A)$ is square-free when $\mathbf{SL}_1(A)$ is a *stably k-rational variety* (i.e. $\mathbf{SL}_1(A) \times_k \mathbb{A}^n_k$ is k-birational to an affine space for an integer n). In this setting, Suslin's conjecture is a special case of the Kneser-Tits problem on R-equivalence. See [Gil2, §2.2] for further details.

I.3 Reductions of the problem

There are some (well-known) reductions of Suslin's Conjecture. First of all, one can restrict to checking Suslin's conjecture for central division algebras as the isomorphism class of $\mathbf{SK}_1(A)$ depends only on the Brauer class of A (and as A is Brauer-equivalent to a unique central division k-algebra by Wedderburn's theorem).

Furthermore, suppose D is a central division k-algebra of $\deg(D) = \mathrm{ind}_k(D) = n$ and let $n = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ be a prime factorisation of n. Then Brauer's decomposition theorem [GS, Prop. 4.5.16] gives central division k-algebras D_i for $i = 1, \ldots, r$ such that $\mathrm{ind}_k(D_i) = p_i^{e_i}$ and such that

$$D \cong D_1 \otimes \ldots \otimes D_r. \tag{I.3}$$

This decomposition induces a decomposition of $\mathbf{SK}_1(D)$: [GS, Ch. 4. Ex. 9 (a)]

$$\mathbf{SK}_1(D) \cong \mathbf{SK}_1(D_1) \oplus \ldots \oplus \mathbf{SK}_1(D_r). \tag{I.4}$$

So in order to verify Suslin's conjecture, one can even restrict to central division algebras of primary degrees.

We can even reduce further and restrict to central division algebras of index p^2 for a prime p. Indeed using the index reduction formula [SVdB, Thm. 1.3], Blanchet gets the following result which justifies this restriction.

Proposition I.14 ([Bla, Prop. 4])

Let A be a central simple k-algebra of $\mathrm{ind}_k(A) = n$. Suppose $r \mid n$, then there exists a field extension K of k such that $\mathrm{ind}_K(A_K) = r$.

Remark I.15 – This proposition would even allow us to restrict to central division algebras of index p^2 without using a Brauer decomposition of the central division algebra. However it would be unfair to withhold the isomorphism (I.4) from the reader's knowledge.

Rehmann-Tikhonov-Yanchevskiĭ prove that one can even restrict to check Suslin's conjecture for cyclic division algebras [RTY, Thm. 0.19], which immediately follows from the following theorem.

Theorem I.16 (ibid., Thm. 0.14)

For any field k there exists a (regular) field extension K such that

(i) any central simple K-algebra is cyclic, and

(ii) for any central simple k-algebra A, $\mathrm{ind}_K(A_K) = \mathrm{ind}_k(A)$.

On the other hand Prokopchuk-Tikhonov-Yanchevskiĭ prove that we can make a restriction to central simple algebras of the form $[(a,b)_p \otimes [(c,d)_p$ [PTY]. This follows by a theorem similar to the previous one.

Theorem I.17 (loc. cit.)

Let A be a central division algebra over a field k with $\mathrm{ind}_k(A) = p^2$. Then there exists a field extension K of k and $a, b, c, d \in K$ such that $\mathrm{ind}_K(A_K) = \mathrm{ind}_k(A)$ and

$$A_K \sim_{\mathrm{Br}} [(a,b)_p \otimes_K [(c,d)_p.$$

Remark I.18 – Note that [PTY] actually only contains an explicit proof of the case $\mathrm{char}(k) \neq p$, but their methods equally work in the case when $\mathrm{char}(k) = p$. As main tool, the proof uses the index reduction formula [SVdB, Thm. 1.3]. In the case $\mathrm{char}(k) \neq p$ and $\mathrm{ind}_k(A) = p^2$, they also explain why (to prove Suslin's conjecture) they can assume k to have a

p-th primitive root of unity so that they can surely define symbol algebras (ibid, p. 2). Let us recall the argument. Suppose $\xi_p \in \overline{k}$ a primitive p-th root of unity and $\xi_p \notin k$ (so in particular p odd). Then $[k(\xi_p) : k] \leq p-1$ as ξ_p is a root of $\sum_{i=0}^{p-1} X^i$. But then $\mathbf{SK}_1(A)(k) \to \mathbf{SK}_1(A)(k(\xi_p))$ is injective (Remark I.13), so that it suffices to prove $\mathbf{SK}_1(A_{k(\xi_p)}) \neq 0$.

So all in the end, we have the following restriction.

Proposition I.19

Suslin's conjecture holds if and only if $\mathbf{SK}_1(A) \neq 0$ for all cyclic division algebras A of the form $[(a,b)_p \otimes [(c,d)_p$.

Merkurjev proves in two different ways that Suslin's conjecture holds for central simple algebras of 2-primary index, i.e. he proves the following theorem.

Theorem I.20 ([Mer1, Mer4])

If A is a central simple k-algebra with $4 \,|\, \mathrm{ind}_k(A)$, then $\mathbf{SK}_1(A) \neq 0$.

He proves this using the reductions above. Actually, he does not need Theorem I.16 or I.17 for this reduction as it is known that any central simple algebra of degree 4 and period 1 or 2 is a product of two quaternion algebras, what is called a *biquaternion algebra* [Alb1, p.369].

I.4 Overview of the thesis

In this thesis we study cohomological invariants of $\mathbf{SK}_1(A)$. It is the hope that these invariants help to describe and understand $\mathbf{SK}_1(A)$ in a better way. Most of the invariants found in the literature are only defined if $\mathrm{ind}_k(A) \in k^\times$.

In Chapter 1, we recall the notion of invariants and cycle modules. We also give an overview of the known invariants of $\mathbf{SK}_1(A)$ and explain why these invariants can explain the examples of non-trivial SK_1.

In Chapter 2, we generalise these invariants to any central simple algebra. This is done by a lift from positive characteristic to characteristic zero. The lift is performed in a generic way, i.e. it does not depend on the definition of the invariants. It rather uses the existence so that given any invariant, we can generalise it to any central simple algebra.

In Chapter 3, we compare the invariants into play. This allows us to prove that an invariant introduced by Kahn is non-trivial for Platonov's examples knowing that another invariant is non-trivial in the same case. For biquaternion algebras, we compare an invariant of Knus-Merkurjev-Rost-Tignol that already exists in characteristic 2 to an invariant obtained in Chapter 2. We also generalise a formula of Merkurjev for the value of the centre of a biquaternion algebra to the tensor product of two symbol algebras.

Chapter 1

Cycle modules and invariants

"Science is a wonderful thing if one does not have to earn one's living at it."
— Albert Einstein

In this chapter we recall some notions needed in the rest of the thesis. For a field k and two functors

$$A : k\text{-}\mathfrak{fields} \to \mathfrak{Sets} \quad \text{and} \quad M : k\text{-}\mathfrak{fields} \to \mathfrak{Sets},$$

a natural transformation of functors $\varphi : A \to M$ is called an *invariant* of A with values in M. So for every field extension K of k, there exists a map $\varphi_K : A(K) \to M(K)$ which is functorial to other field extensions, i.e. if K' is a field extension of K, we have a commutative diagram

$$\begin{tikzcd}
A(K) \arrow[r, "\varphi_K"] \arrow[d] & M(K) \arrow[d] \\
A(K') \arrow[r, "\varphi_{K'}"'] & M(K'),
\end{tikzcd}$$

where the vertical maps are coming from the functors A and M. In our results, we do not work with the 'vague' category of sets. Our functors have values in the more concrete category of groups (or abelian groups). So let

$$A : k\text{-}\mathfrak{fields} \to \mathfrak{Groups} \quad \text{and} \quad M : k\text{-}\mathfrak{fields} \to \mathfrak{Groups}$$

be two *group functors*. By an invariant φ of A in M, we mean a natural transformation of functors as before, but we also require for every field extension K of k, the morphism φ_K to be a group morphism. If M even has values in $\mathfrak{Ab}$, all invariants of A in M form an abelian group $\mathrm{Inv}(A, M)$. When M is (some kind of) a cohomology group, we say φ is a *cohomological invariant* of A.

Merkurjev introduces a nice framework to work with [Mer3, §2]. He rather considers M as (a component of) a cycle module and then gives a practical alternative description of invariants when A is an algebraic group. In this chapter, we recall the formalism of Rost's cycle modules [Ros2, §1,2] and Merkurjev's description. Using this setting, we recall the various invariants of $\mathbf{SK}_1$ found in the literature. We first give some introductory examples of cohomology groups we use later on. These lead us to the formal definition of a cycle module.

1.1 Cohomology groups

In this section, we take F to be a field and $m > 0$ an integer invertible in F.

(a) Definition – Let $\mu_m^{\otimes i}$ be the i-th tensor product of μ_m as $\mathbb{Z}/m\mathbb{Z}$-module ($i \geq 0$). Then consider the following Galois cohomology groups.

Definition 1.1

For any field F and integers $i, m \geq 0$ with $m \in F^\times$, we define

$$H_m^i(F) = H^i(F, \mu_m^{\otimes i}(-1)) \qquad \text{with} \qquad \mu_m^{\otimes i}(-1) = \mathrm{Hom}_{\Gamma_F}(\mu_m, \mu_m^{\otimes i}),$$

a Tate twist. For $i < 0$, we set $H_m^i(F) = 0$.

Clearly, $\mu_m^{\otimes i+1}(-1) = \mu_m^{\otimes i}$ for all $i \geq 0$, and so $H_m^{i+1}(F) = H^{i+1}(F, \mu_m^{\otimes i})$.[1] The short exact Kummer sequence

$$1 \to \mu_m \to F_s^\times \xrightarrow{m} F_s^\times \to 1 \tag{1.1}$$

then implies the well-known cohomological interpretation of the part of m-torsion of the Brauer group of F:

$$_m\mathrm{Br}(F) \cong H_m^2(F). \tag{1.2}$$

[1] We try to use as much as possible the superscript $i+1$ in stead of i to keep up with tradition (which rather defines $H_m^i(F)$ as $H^i(F, \mu_m^{\otimes i})$) and to stay in conformity with the wild case (§2.2.1) where it is clearly more natural to use this superscript. In any case, any appearance of $H_m^i(F)$ is to be interpreted as the Galois cohomology group defined over here (and not as $H^i(F, \mu_m^{\otimes i})$ - unless $\mu_m \subset F$).

(b) $K_n(F)$-module structure – Consider *Milnor's K-groups*[2] $K_n(F)$ for an integer $n \geq 0$. Recall that

$$K_n(F) = \underbrace{F^\times \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F^\times}_{n \text{ times}}/J,$$

where J is the subgroup generated by the symbols of the form $x_1 \otimes \ldots \otimes x_n$ such that $x_i + x_j = 1$ for some $1 \leq i < j \leq n$. The primitive symbols $x_1 \otimes \ldots \otimes x_n$ are denoted as $\{x_1, \ldots, x_n\}$. Kummer's short exact sequence (1.1) induces an isomorphism $h^1_{m,F}$ as composition $K_1(F)/mK_1(F) = F^\times/(F^\times)^m \cong H^1(F, \mu_m)$. We retrieve the *Galois symbol* using the cup-product:

$$\begin{aligned} h^n_{m,F} : K_n(F)/mK_n(F) &\rightarrow H^n(F, \mu_m^{\otimes n}), \quad \text{defined by} \\ \{x_1, \ldots, x_n\} &\mapsto h^1_{m,F}(x_1) \cup \ldots \cup h^1_{m,F}(x_n). \end{aligned} \tag{1.3}$$

As a matter of fact, $h^n_{m,F}$ is an isomorphism (Bloch-Kato conjecture - theorem of Voevodsky-Rost-Weibel [BK, Voe, Ros3, Wei2]). We call this the *Bloch-Kato isomorphism*. By taking the cup product with this Galois symbol, we can define a $K_n(F)$-module structure on $(H^{i+1}_m(F))_{i \geq 0}$:

$$K_n(F) \times H^{i+1}_m(F) \rightarrow H^{n+i+1}_m(F) : (a, b) \mapsto h^n_{m,F}(\bar{a}) \cup b.$$

We denote this scalar product by $a \cdot b = h^n_{m,F}(\bar{a}) \cup b$ for $a \in K_n(F)$, $\bar{a}$ its class in $K_n(F)/mK_n(F)$, and $b \in H^{i+1}_m(F)$.

> *Remark* 1.2 – Suppose F contains an m-th primitive root of unity so that $H^i_m(F) \cong H^i(F, \mu_m^{\otimes i})$. Then under the isomorphism (1.2), the class of a symbol F-algebra $(a, b)_m$ is mapped to $h^2_{m,F}(\{a, b\})$ [GS, Prop. 4.7.1].

(c) Residue maps – Suppose F is complete for a discrete valuation v. The valuation v extends uniquely to a valuation on F_s, which in its turn gives rise to a residue morphism $\Gamma_F \rightarrow \Gamma_{\kappa(v)}$ of absolute Galois groups. This induces for any integer $i \geq 0$ an injection

$$\varphi_i : H^i_m(\kappa(v)) \rightarrow H^i_m(F).$$

[2] In the following, we mainly use Milnor K-groups. To ease notations, we do not use the superscript M of the more common notation $K^M_n(F)$ of Milnor K-groups. When using *Quillen K-groups*, we use the notation K^Q_n.

Furthermore, if π is a uniformiser with respect to v, we have a map for any $i \geq 0$:

$$\psi_i : H^i_m(\kappa(v)) \to H^{i+1}_m(F) : a \mapsto h^1_{m,F}(\pi) \cup \varphi_i(a).$$

It can be proved that $\varphi_{i+1} \oplus \psi_i$ is an isomorphism [GMS, Prop. 7.7]. Hence this gives us a morphism $\partial^{i+1}_v : H^{i+1}_m(F) \to H^i_m(\kappa(v))$, called a *residue morphism*. So we have a split exact sequence

$$0 \to H^{i+1}_m(\kappa(v)) \to H^{i+1}_m(F) \xrightarrow{\partial^{i+1}_v} H^i_m(\kappa(v)) \to 0. \tag{1.4}$$

Suppose F is endowed with a discrete valuation v, but is not complete for the topology defined by v. Then we still have a residue. Indeed, take $\hat{F}$ to be the completion of F with respect to v, which also has residue field $\kappa(v)$. The residue is then defined as composition

$$\partial^{i+1}_v : H^{i+1}_m(F) \to H^{i+1}_m(\hat{F}) \to H^i_m(\kappa(v)),$$

where obviously the last morphism is the residue for the complete field $\hat{F}$.

We refer to [Ser1, Ch. II & III] for the assertions on valuation theory.

Remark 1.3 – These notions can be extended to other Galois cohomology groups of fields with a discrete valuation. There exists for example in general a short exact sequence as (1.4) for the Galois cohomology groups $H^i(F, \mu_n^{\otimes i+j})$ for any integer j. They are defined in a similar way. See [GMS, §7] for more information on these residue maps.

(d) Relative version – We define a relative version of the Galois cohomology groups $H^{i+1}_m(F)$.

Definition 1.4

Let A be a central simple F-algebra with $\mathrm{ind}_F(A) = n \in F^\times$ and with Brauer class $[A] \in {}_n\mathrm{Br}(F) \cong H^2_n(F)$. Then define for any integers $i \geq 1$ and r

$$H^{i+1}_{n,A^{\otimes r}}(F) = H^{i+1}_n(F) / \left(H^{i-1}(F, \mu_n^{\otimes i-1}) \cup r[A] \right).$$

Remark 1.5 – Note that if $r \equiv 0 \mod \mathrm{per}_k(A)$, we find $H^{i+1}_{n,A^{\otimes r}}(F) = H^{i+1}_n(F)$ as $r[A] = 0$ in $\mathrm{Br}(F)$. We could hence restrict the possible values of r, but for ease of notation we just take r any integer. Allowing the case $r \equiv 0 \mod \mathrm{per}_k(A)$ to happen, we cover both the relative and the absolute version with the relative one.

Remark 1.6 – Remark also that by the Bloch-Kato isomorphism and the $K_n(F)$-module-structure, we can give an equivalent definition:

$$H^{i+1}_{n,A^{\otimes r}}(F) = H^{i+1}(F, \mu_n^{\otimes i}) / \left(K_{i-1}(F) \cdot r[A]\right). \tag{1.5}$$

If F is complete for a discrete valuation v, we can extend the residues of $H^{i+1}_n(F)$ to relative residues. We suppose A to be a central simple $\kappa(v)$-algebra with $\mathrm{ind}_{\kappa(v)}(A) \in \kappa(v)^\times$ and $\mathrm{ind}_{\kappa(v)}(A) = n \in F^\times$.

Under the injection ${}_n\mathrm{Br}(\kappa(v)) \to {}_n\mathrm{Br}(F)$ from (1.4), the class of A maps to the class of a central simple K-algebra B_K, called a *lifted central simple algebra*. In §2.1.2 (a) we give more comments on this construction.[3] The description in terms of explicit cocycles [GMS, Ex. 7.12] guarantees that

$$\partial_v^{i+1}(H^{i-1}(F, \mu_n^{\otimes i-1}) \cup r[B_K]) \subset H^{i-2}(\kappa(v), \mu_n^{\otimes i-2}) \cup r[A].$$

Then we get a commutative diagram (for $i \geq 2$):

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}(\kappa(v), \mu_n^{\otimes i-1}) \arrow[r] \arrow[d, "\cup r[A]"] & H^{i-1}(F, \mu_n^{\otimes i-1}) \arrow[r] \arrow[d, "\cup r[B_K]"] & H^{i-2}(\kappa(v), \mu_n^{\otimes i-2}) \arrow[r] \arrow[d, "\cup r[A]"] & 0 \\
0 \arrow[r] & H^{i+1}(\kappa(v), \mu_n^{\otimes i}) \arrow[r] & H^{i+1}(F, \mu_n^{\otimes i}) \arrow[r] & H^{i}(\kappa(v), \mu_n^{\otimes i-1}) \arrow[r] & 0.
\end{tikzcd}$$

As the short exact sequences are split, the snake lemma allows us to construct the following short exact sequence:

$$0 \to H^{i+1}_{n,A^{\otimes r}}(\kappa(v)) \to H^{i+1}_{n,B_K^{\otimes r}}(F) \xrightarrow{\partial^{i+1}_{v,A^{\otimes r}}} H^{i}_{n,A^{\otimes r}}(\kappa(v)) \to 0. \tag{1.6}$$

The map $\partial_{v,A^{\otimes r}}$ is the relative residue. Furthermore, as (1.4) is split, (1.6) is so too.

[3] We use the subscript K in B_K as this is in conformity with the discussion in §2.1.2 (a), where we pass via Azumaya algebras.

1.2 Cycle modules

The common properties of $H_n^{i+1}(F)$ and Milnor K-groups have inspired Rost to define a formal structure respecting these homological properties [Ros2, §§1,2]. Let us briefly recall this formalism of cycle modules.

(a) Definition of a cycle module – For a discrete valuation ring R, let R-$\mathfrak{fields}$ be the category of R-fields; these are R-algebras which are fields, so field extensions of Frac(R) or $\kappa(v)$, the residue field. Let us literally recall the definition of a cycle module.

Definition 1.7 (loc. cit.)

For any discrete valuation ring R, a *cycle module M with base R* consists of an object function

$$R\text{-}\mathfrak{fields} \to \mathfrak{Ab}$$

equipped with a grading $M = (M_j)_{j\geq 0}$ and data D1-D4 satisfying compatibility (R1a-R3e) and geometrical rules (FD and C) as below: (E, F objects in R-$\mathfrak{fields}$ and φ a morphism in R-$\mathfrak{fields}$)

D1: Any $\varphi : F \to E$ induces $\varphi_* : M(F) \to M(E)$ of degree 0.

D2: Any finite $\varphi : F \to E$ induces $\varphi^* : M(E) \to M(F)$ of degree 0.

D3: For all F, the group $M(F)$ has a $K_n(F)$-module structure such that $K_n(F) \cdot M_m(F) \subset M_{n+m}(F)$ ($n, m \geq 0$ integers).

D4: If F is an R-field with a discrete valuation v such that the residue field $\kappa(v)$ is also a R-field, then there exists a *residue* $\partial_v : M(F) \to M(\kappa(v))$ of degree -1.

Remark 1.8 – Note that for obtaining his goals, Rost puts more restrictions on his base R, but he comments it is allowed to moderate these (ibid., §1, p. 328). Also, in loose notation M_j for $j < 0$ equals the trivial group. A morphism from a graded abelian group $(A_j)_{j\geq 0}$ to a graded abelian group $(B_j)_{j\geq 0}$ is a collection of group morphism $\varphi_j : A_j \to B_{j+d}$ for a fixed integer d, the *degree* of the morphism.

Let us now give the rules mentioned in the definition. In all of this, let E, F, G be arbitrary R-fields and suppose that any map between fields is a morphism in R-$\mathfrak{fields}$. For a discrete valuation on an R-field, we assume that the residue field is also an R-field.

R1a: Any $\varphi : F \to E, \psi : E \to G$ satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

R1b: Any finite $\varphi : F \to E, \psi : E \to G$ satisfy $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

R1c: Take $\varphi : F \to E, \psi : F \to G$ with φ finite and $S = G \otimes_F E$. For any $p \in \mathrm{Spec}(S)$, let $\varphi_p : G \to S/p, \psi_p : E \to S/p$ be the natural maps and let l_p be the length of the localised ring S_p. Then,

$$\psi_* \circ \varphi^* = \sum_p l_p \cdot (\varphi_p)^* \circ (\psi_p)_*.$$

R2: For $\varphi : F \to E, x \in K_*F, y \in K_*E, \rho \in M(F), \mu \in M(E)$, one has (with φ finite in R2b and R2c):

R2a: $\varphi_*(x \cdot \rho) = \varphi_*(x) \cdot \varphi_*(\rho)$,

R2b: $\varphi^*(\varphi_*(x) \cdot \mu) = x \cdot \varphi^*(\mu)$, and

R2c: $\varphi^*(y \cdot \varphi_*(\rho)) = \varphi^*(y) \cdot \rho$.

R3a: Let $\varphi : E \to F$ and let v be a discrete valuation on F which restricts to a non-trivial valuation w on E with ramification index e. Let $\bar{\varphi} : \kappa(w) \to \kappa(v)$ be the induced map. Then,

$$\partial_v \circ \varphi_* = e \cdot \bar{\varphi}_* \circ \partial_w.$$

R3b: Let $\varphi : F \to E$ be finite and v a discrete valuation on F. For any extension w of v on E, let $\varphi_w : \kappa(v) \to \kappa(w)$ be the induced map. Then,

$$\partial_v \circ \varphi^* = \sum_{w|v} \varphi_w^* \circ \partial_w.$$

R3c: Let $\varphi : E \to F$ and let v be a discrete valuation on F which is trivial on E. Then,

$$\partial_v \circ \varphi_* = 0.$$

R3d: Let $\varphi : E \to F$, let v be a valuation on F which is trivial on E, let $\bar{\varphi} : E \to \kappa(v)$ be the induced map, and let π be an uniformiser of v. Define furthermore $s_v^\pi : M(F) \to M(\kappa(v))$ by $s_v^\pi(\rho) = \partial_v(\{-\pi\} \cdot \rho)$, then

$$s_v^\pi \circ \varphi_* = \bar{\varphi}_*.$$

R3e: Let v be a discrete valuation on F, u a v-unit, and $\rho \in M(F)$, then

$$\partial_v(\{-u\} \cdot \rho) = -\{\bar{u}\} \cdot \partial_v(\rho).$$

For any R-scheme $\mathcal{X}$, we denote $M(x) = M(\kappa(x))$ for $x \in \mathcal{X}$ with residue field $\kappa(x)$. If $\mathcal{X}$ is irreducible, we denote its generic point by ξ. If $\mathcal{X}$ is normal, any $x \in \mathcal{X}^{(1)}$ induces $\partial_x : M(\xi) \to M(x)$. For $x, y \in \mathcal{X}$, we define ∂^x_y. One sets $\partial^x_y = 0$ if $Z = \overline{\{x\}}$ and $y \notin Z^{(1)}$. Otherwise, let $\tilde{Z} \to Z$ be the normalisation and

$$\partial^x_y = \sum_{z|y} \varphi^*_z \circ \partial_z,$$

where z runs through all points of $\tilde{Z}$ lying above y and where φ_z is the finite morphism $\kappa(y) \to \kappa(z)$.

FD: *(Finite support of divisors)* Let $\mathcal{X}$ be a normal R-scheme and $\rho \in M(\xi)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in \mathcal{X}^{(1)}$.

C: *(Closedness)* Let $\mathcal{X}$ be an integral R-scheme, local of dimension 2 and let x_0 be its closed point. Then,

$$0 = \sum_{x \in \mathcal{X}^{(1)}} \partial^x_{x_0} \circ \partial^\xi_x : M(\xi) \to M(x_0).$$

(b) The base and coexistence of two cycle modules – In the classical case, a cycle module has as base a field (with definition as above replacing R by a field). In this thesis however, we use cycle modules with a complete discrete valuation ring R as base. Let K be the fraction field of R and k its residue field. A cycle module M with base R attaches then to any field extension L of K a graded group $M(L)$ and likewise, to any field extension $\overline{L}$ of k a graded group $M(\overline{L})$.

Remark that one can hence restrict a cycle module with base R to a cycle module with base K and to one with base k, by restricting either to field extensions of K or to field extensions of k. A cycle module with base R is therefore the coexistence of two cycle modules with as base a field with an additional link given by the data D1-D4 (in the mixed characteristic case only D4). So we use the notion of a cycle module with base R on the one hand to ease notation and on the other hand to work in a more general setting. Nevertheless, one could reformulate the arguments using two different cycle modules and using the link given by the data as an additional link of the two cycle modules.

(c) Gersten complex – Take as above R any complete discrete valuation ring with fraction field K and residue field k. Let F be an R-field, X an F-variety, and M a cycle module. The existence of residues (D4) and the rules of cycle modules induce a cycle complex, called the *Gersten complex* $C_*(X, M_j)$ [Ros2, §3.3] ($i, j \geq 0$):

$$\ldots \to \bigoplus_{x \in X^{(i-1)}} M_{j-i+1}(F(x)) \overset{\partial^{i-1}}{\to} \bigoplus_{x \in X^{(i)}} M_{j-i}(F(x)) \overset{\partial^{i}}{\to} \bigoplus_{x \in X^{(i+1)}} M_{j-i-1}(F(x)) \to \ldots,$$

where $F(x)$ is the residue field of x, a point of codimension i. The map ∂^i is the sum of the residues induced by the valuations associated with the codimension 1 points of $X^{(i)}$. The homology of this complex on spot i is denoted $A^i(X, M_j)$.

(d) Privileged examples – Let us link these cycle modules to the previous section of Galois cohomology groups. Let R be a complete discrete valuation ring with fraction field K and residue field k, let A be a central simple k-algebra of $\mathrm{ind}_k(A) = n$ such that $n \in K^\times$ and $n \in k^\times$, and let B_K be a lifted central simple K-algebra. Then the functors

$$\mathcal{H}^*_m = (\mathcal{H}^i_m)_{i \geq 0} : R\text{-}\mathfrak{fields} \to \mathfrak{Ab} \quad : \quad F \mapsto \big(H^i_m(F)\big)_{i \geq 0} \text{ and}$$

$$\mathcal{H}^*_{n,\mathcal{B}^{\otimes r}} = (\mathcal{H}^i_{n,\mathcal{B}^{\otimes r}})_{i \geq 2} : R\text{-}\mathfrak{fields} \to \mathfrak{Ab} \quad : \quad F \mapsto \Big(H^i_{n,B^{\otimes r}}(F)\Big)_{i \geq 2}$$

are cycle modules where r is any integer and $H^i_{n,B^{\otimes r}}(F)$ is to be interpreted in the appropriate way. For a field extension F of k, it is $H^i_{n,A^{\otimes r}}(F)$. For a field extension F of K, it is rather $H^i_{n,B_F^{\otimes r}}(F)$ with $B_F = B_K \otimes_K F$. If we restrict $\mathcal{H}^*_{n,\mathcal{B}^{\otimes r}}$ to field extensions of k (resp. K) as in §1.2 (b), we write it as $\mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}$ (resp. $\mathcal{H}^*_{n,\mathcal{B}_K^{\otimes r}}$).

The verification of the rules R1a-R3e, FD, and C for $\mathcal{H}^*_m$ in the equicharacteristic case was done by Rost (ibid., Rem. 1.11). The case of mixed characteristics follows analogously. This also induces $\mathcal{H}^*_{n,\mathcal{B}^{\otimes r}}$ to be a cycle module as the data and rules of $\mathcal{H}^*_m$ behave well under taking the quotients into play (see e.g. (1.6)). For R-fields endowed with a valuation but not complete, the residue for $\mathcal{H}^*_{n,\mathcal{B}^{\otimes r}}$ is retrieved by passing via a completion (as in §1.1 (c)).

Other examples of cycle modules with as base a discrete valuation ring R (or possibly just a field) are Milnor's K-groups $(K_i)_{i\geq 0}$. Datum $D1$ is defined in the obvious way. Let E be a finite field extension of an R-field F, then datum D2 is induced by the norm $N_{E/F}$ applied to the primitive symbols [BT, Ch. I, §5]. Datum D3 is defined by the multiplicative structure of the K-groups:

$$K_n(F) \times K_m(F) \mapsto K_{n+m}(F): \quad \text{defined by}$$
$$(\{x_1, \dots, x_n\}, \{y_1, \dots, y_m\}) \mapsto (\{x_1, \dots, x_n, y_1, \dots y_m\}).$$

Now let F be an R-field with a discrete valuation v, then the residue $K_n(F) \to K_{n-1}(\kappa(v))$ – datum D4 – is defined by

$$\begin{aligned} \{\pi, x_2, \dots, x_n\} &\mapsto \{\bar{x}_2, \dots, \bar{x}_n\}, \\ \{x_1, x_2, \dots, x_n\} &\mapsto 0, \end{aligned}$$

with $x_1, \dots, x_n \in \mathcal{O}_v^\times$ and π an uniformiser of F [Mil5, Lem. 2.1].

Furthermore if $r >$ is an integer, then $(K_i/r)_{i\geq 0}$ also forms a cycle module with base R as the definitions above go through. If r is prime to the characteristic of the residue field of R (and hence also to the characteristic of the fraction field of R), we have a short exact sequence similar to (1.4). Indeed in that case for any R-field F complete for a discrete valuation v, there is a short exact sequence for any integer $i \geq 0$ (ibid., Lem. 2.6):

$$0 \to K_{i+1}(\kappa(v))/r \xrightarrow{i} K_{i+1}(F)/r \xrightarrow{\partial_v^{i+1}} K_i(\kappa(v))/r \to 0. \tag{1.7}$$

Here, ∂_v^{i+1} is of course the residue as above and i is defined by

$$\{\bar{x}_0, \dots, \bar{x}_i\} \pmod r \mapsto \{x_0, \dots, x_i\} \pmod r,$$

for $x_0, \dots, x_i \in \mathcal{O}_v^\times$. Note that this sequence is split by the retraction $\psi : K_i(\kappa(v))/r \to K_{i+1}(F)/r$ defined by

$$\{\bar{x}_1, \dots, \bar{x}_i\} \pmod r \mapsto \{\pi, x_1, \dots, x_i\} \pmod r,$$

where π is still the uniformiser as above. Note that by the Bloch-Kato isomorphism, this comes down to the short exact sequence for the $H^i(k, \mu_n^{\otimes i})$'s (as in Remark 1.3). The similar behaviour of both groups was actually a motivation to believe in the Bloch-Kato conjecture.

1.3 Invariants à la Merkurjev

In this section, let k be a field and $M = (M_j)_{j\geq 0}$ a cycle module with base k and of bounded exponent (i.e. $rM = 0$ for some integer r). Merkurjev discovered a interesting deep link between the groups $A^0(\mathbf{G}, M_j)$ and invariants of an algebraic k-group $\mathbf{G}$ in M of degree j. We recall this link, but first we give the notion of the degree of an invariant with values in a cycle module.

(a) Invariants with values in cycle modules – Suppose $\mathbf{G} : k\text{-}\mathfrak{fields} \to \mathfrak{Groups}$ is a group functor (e.g. an algebraic group) and consider furthermore M_j (for an integer $j \geq 0$) as group functor $k\text{-}\mathfrak{fields} \to \mathfrak{Groups}$. An *invariant ρ of $\boldsymbol{G}$ in M of degree j* is an invariant $\rho : \mathbf{G} \to M_j$. These invariants form an abelian group, which we denote by $\operatorname{Inv}^j(\mathbf{G}, M)$. We can define the same terminology if M is any functor of graded abelian groups.

(b) Merkurjev's link – Let $\mathbf{G}$ be an algebraic group, then Merkurjev constructs an injective morphism

$$\theta : \operatorname{Inv}^j(\mathbf{G}, M) \to A^0(\mathbf{G}, M_j) : \rho \mapsto \rho_K(\xi), \tag{1.8}$$

where $K = k(\mathbf{G})$ and $\xi \in \mathbf{G}(K)$ is the generic point of $\mathbf{G}$. He proves that the image is the *multiplicative subgroup* $A^0(\mathbf{G}, M_j)_{\text{mult}}$ consisting of the *multiplicative elements* of $A^0(\mathbf{G}, M_j)$ [Mer3, Lem. 2.1 and Thm. 2.3]. These are the elements $x \in A^0(\mathbf{G}, M_j)$ such that

$$p_1^*(x) + p_2^*(x) = m^*(x),$$

where p_1^*, p_2^*, and m^* are the morphisms $A^0(\mathbf{G}, M_j) \to A^0(\mathbf{G} \times \mathbf{G}, M_j)$ induced by the two projections $p_1, p_2 : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the multiplication $m : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$.

He also proves that $A^0(\mathbf{G}, M_j)_{\text{mult}} \subset \tilde{A}^0(\mathbf{G}, M_j)$, where $\tilde{A}^0(\mathbf{G}, M_j)$ is the *reduced subgroup* of $A^0(\mathbf{G}, M_j)$ (ibid., Lem. 1.9). The reduced subgroup is the kernel of the morphism $u^* : A^0(\mathbf{G}, M_j) \to A^0(\mathbf{1}, M_j)$ induced by the unit morphism $u : \mathbf{1} \to \mathbf{G}$. This morphism u^* also induces a splitting $A^0(\mathbf{G}, M_j) \cong \tilde{A}^0(\mathbf{G}, M_j) \oplus A^0(k, M_j)$, whence the equivalent definition:

$$\tilde{A}^0(\mathbf{G}, M_j) = A^0(\mathbf{G}, M_j)/A^0(k, M_j);$$

i.e. “$A^0(\mathbf{G}, M_j)$ modulo the constants”.

(c) What about $\boldsymbol{SK}_1$? – So, we would like to describe invariants of $\mathbf{SK}_1(A)$ using (1.8). However $\mathbf{SK}_1(A)$ is not an algebraic group. But for any field extension F of k, we do have a canonical projection $\mathbf{SL}_1(A)(F) \to \mathbf{SL}_1(A)(F)/[A_F^\times, A_F^\times] \cong \mathbf{SK}_1(A)(F)$ which gives us an injective morphism on invariants.

Lemma 1.9

Let k be a field, A a central simple k-algebra, and M a cycle module. The projection of k-functors $\pi : \mathbf{SL}_1(A) \to \mathbf{SK}_1(A)$ induces for any integer j an injection

$$\tilde{\pi} : \mathrm{Inv}^j(\mathbf{SK}_1(A), M) \hookrightarrow \mathrm{Inv}^j(\mathbf{SL}_1(A), M).$$

This lemma allows us to use Merkurjev's description when working with invariants of $\mathbf{SK}_1(A)$. We just look at the induced invariant for $\mathbf{SL}_1(A)$.

1.4 Invariants of SK_1

In order to explain Platonov examples of non-trivial $\mathbf{SK}_1$, Suslin conjectured in 1991 the existence of an invariant for any central simple k-algebra A of $\mathrm{ind}_k(A) = n \in k^\times$ [Sus, Conj. 1.16]:

$$\rho_A \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}}). \tag{1.9}$$

Here we consider $\mathcal{H}^*_{n,\mathcal{A}} = (\mathcal{H}^i_{n,A})_{i\geq 2}$ as a cycle module with base k. Making the right hypotheses on A, we could see it as a cycle module with as base a complete discrete valuation ring R restricted to its fraction field or residue field as in §1.2 (b).

(a) Suslin 1991 – Let us explain why Suslin conjectured the existence of such an invariant. So we use now the same notation as in Example I.10. In this case $\mathrm{SK}_1(A)$ can be expressed in terms of Brauer groups, i.e. second Galois cohomology groups. On the other hand, F is a field equipped with a discrete valuation of rank 2, so this induces the existence of two residues $\partial^3_{t_1}, \partial^4_{t_2}$ in Galois cohomology (§1.1 (c) & (d)). Then using (I.2), the invariant

should be able to complete the diagram:

$$\begin{tikzcd} \mathrm{SK}_1(A) \arrow[r, "\cong"] \arrow[d, "\rho_{A,F}"'] & \mathrm{Br}(K/k)/(\mathrm{Br}(K_1/k)\mathrm{Br}(K_2/k)) \arrow[d] \\ H^4_{n^2,A}(F) \arrow[r, "\partial^3_{t_1}\circ\partial^4_{t_2}"'] & H^2_{n^2}(k)/\partial^3_{t_1}\circ\partial^4_{t_2}(H^2(k,\mu_{n^2}^{\otimes 2})\cup[A]). \end{tikzcd} \tag{1.10}$$

In 1991, Suslin was not able to define this invariant in full generality. He was however able to define an invariant

$$\rho_{\mathrm{S91},A} \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes 2}}),$$

satisfying a compatibility as above. In particular, this invariant is not trivial for Platonov's examples (see also proof of Theorem 3.16).

(b) Biquaternion algebras – In the case of biquaternion algebras, Rost was able to define a related invariant of $\mathbf{SK}_1(A)$. Suppose $A = (a,b)\otimes(c,d)$ is a biquaternion algebra over a field k of $\mathrm{char}(k) \neq 2$. Then *Rost's invariant* $\rho_{\mathrm{Rost},A}$ is an invariant sitting in $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_2)$ [Mer2, Thm. 4]. Moreover, it fits into an exact sequence:

$$0 \to \mathbf{SK}_1(A)(k) \to H^4(k, \mathbb{Z}/2\mathbb{Z}) \to H^4(k(Y), \mathbb{Z}/2\mathbb{Z}), \tag{1.11}$$

where Y is a quadratic k-form defined by

$$ax_1^2 + bx_2^2 - abx_3^2 - cx_4^2 - dx_5^2 + cdx_6^2, \tag{1.12}$$

a so-called *Albert form of A*. Note that $\mu_2^{\otimes i} \cong \mathbb{Z}/2$ as Γ_k-modules for any integer i, which is used freely above (and in the following).

This invariant was generalised in [KMRT, §17] to biquaternion algebras in any characteristic, using Witt groups and Witt rings. The exact definition of this generalisation requires more terminology to be introduced, but after all the definition is very concrete. This contrasts sharply with the other invariants into play, which are defined using (a lot of) homological arguments and which are very abstract by definition. We come back to this generalised invariant in Chapter 3 where we also recall Witt groups and Witt rings.

(c) Suslin 2006 – Using Voevodsky's motivic étale cohomology, Suslin was able to define his conjectured invariant (1.9) in 2006. It is however not clear whether (1.10) commutes for this invariant. We denote this invariant by $\rho_{\mathrm{S06},A}$. It is clear that this invariant (as well as any other invariant) is trivial after base extension to the function field of the Severi-Brauer variety $X = \mathrm{SB}(A)$. Indeed,

$$\begin{tikzcd}
\mathbf{SK}_1(A)(k) \arrow[r] \arrow[d] & H^4_{n,A}(k) \arrow[d] \\
\mathbf{SK}_1(A)(k(X)) \arrow[r] & H^4_{n,A}(k(X))
\end{tikzcd}$$

commutes by definition of an invariant and furthermore $\mathbf{SK}_1(A)(k(X)) = 0$ as $k(X)$ is a splitting field of A (see e.g. [GS, §5.4]).

Suslin also proves his invariant is essentially the same as Rost's invariant $\rho_{\mathrm{Rost},A}$ for a biquaternion algebra A over a field k of $\mathrm{char}(k) \neq 2$. He does this by proving

$$\begin{tikzcd}
\mathbf{SK}_1(A)(k) \arrow[r, "\rho_{\mathrm{S06}}"] \arrow[d, "\mathrm{id}"'] & \ker\big[H^4_{4,A}(k) \to H^4_{4,A}(k(X))\big] \arrow[d, "r_A"] \\
\mathbf{SK}_1(A)(k) \arrow[r, "\rho_{\mathrm{Rost}}"'] & \ker\left[(H^4_2(k) \to H^4_2(k(Y))\right],
\end{tikzcd} \tag{1.13}$$

is a commutative diagram, where r_A is the morphism induced on Galois cohomology by the map $\mu_4^{\otimes 3} \to \mu_2 : a \mapsto a^2$ and where X and Y are as above. This also proves ρ_{S06} is injective for biquaternion algebras and

$$\mathbf{SK}_1(A)(k) \cong \ker\big[H^4_{4,A}(k) \to H^4_{4,A}(k(X))\big].$$

Note that these statements are functorial, so that we can also generalise them to any field extension of k.

(d) Kahn's approach – Kahn revisited Suslin's construction and generalised Suslin's invariant ρ_{S06} [Kah3, §8.B]. For any central simple k-algebra with $n = \mathrm{ind}_k(A) \in k^\times$, he defined for $r = 1, \ldots, \mathrm{per}_k(A) - 1$

$$\rho_r \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^4_{n,\mathcal{A}^{\otimes r}}).$$

Suslin's invariant ρ_{S06} is retrieved setting $r = 1$. It is however not clear whether ρ_{S91} equals ρ_2. Kahn also proves ρ_r is trivial after base extension to the function field of the the generalised Severi-Brauer variety $\mathrm{SB}(r, A)$.

He also gives a bound on the torsion of these invariants as elements of $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$ if $l = \mathrm{per}_k(A)$ is a prime. Indeed from (ibid., Thm. 7.1(c) & Cor. 12.10) it follows that the ρ_r have

- l-torsion if $\mathrm{ind}_k(A) = \mathrm{per}_k(A) = l > 2$,
- l^2-torsion if $\mathrm{ind}_k(A) > \mathrm{per}_k(A) = l > 2$, and
- 2-torsion if $\mathrm{per}_k(A) = 2$.

For any integer n with prime factorisation $p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$, we denote by $\overline{n}$ the integer $p_1^{e_1-1} \cdot \ldots \cdot p_r^{e_r-1}$. If A is a central simple k-algebra A with $n = \mathrm{ind}_k(A) \in k^\times$ and $\mathrm{per}_k(A) = n/\overline{n}$, then we get a similar bound on the torsion using a Brauer decomposition. Take a prime factorisation $n = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ and let $D_1 \otimes \ldots \otimes D_r$ be a Brauer decomposition of A as in (I.3). Then put $m = p_1^{f_1} \cdot \ldots \cdot p_r^{f_r}$, where $f_i = 1$ if $p_i = 2$ or if $\mathrm{ind}_k(D_i) = \mathrm{per}_k(D_i) = p_i > 2$, and $f_i = 2$ if $\mathrm{ind}_k(D_i) > \mathrm{per}_k(D_i) = p_i > 2$. Then it is clear that ρ_r has m-torsion.

On the other hand, Kahn also approaches invariants à la Merkurjev. By calculations with Quillen's K-theory, he shows $A^0(\mathbf{SL}_1(A), \mathcal{H}^4_n)_{\mathrm{mult}}$ is a finite cyclic group [Kah3, Def. 11.3]. So by (1.8) and Lemma 1.9, we also find $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_n)$ to be a finite cyclic group. Using Kahn's calculations (loc. cit.), we can pick a canonical generator that we call *Kahn's invariant* $\rho_{\mathrm{Kahn},A}$ of $\mathbf{SK}_1(A)$.

Furthermore Kahn argues that the size of $\mathrm{Inv}^4(\mathbf{SL}_1(A), \mathcal{H}^*_n)$ is bounded by $\mathrm{ind}(A)/l$ if $n = \mathrm{ind}_k(A)$ is the power of a prime l (ibid., Lem. 12.1). Hence the same holds for $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_n)$ by Lemma (1.9). Using Brauer's decomposition theorem (I.3), it is easy to generalise this statement.

Lemma 1.10

Let k be a field and A a central simple algebra of $\mathrm{ind}_k(A) = n \in k^\times$. Then

$$|\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_n)| \leq \overline{n}.$$

Proof. Let $p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ be a prime decomposition of n and $D_1 \otimes \ldots \otimes D_r$ a Brauer decomposition as in (I.3). Recall that this gives rise to a decomposition of $\mathbf{SK}_1(A)$ (I.4) and that $\mathbf{SK}_1(D_i)$ has $p_i^{e_i}$-torsion [Dra, §23, Lem. 3]. Then the result follows immediately from the primary result of Kahn and the isomorphism

$$H_n^4(k) \cong H^4_{p_1^{e_1}}(k) \oplus \ldots \oplus H^4_{p_r^{e_r}}(k).$$

■

Remark 1.11 – As Kahn mentions, this bound is sharp for biquaternion division algebras [Kah3, §12]. This follows from [Mer3, Prop. 4.9 & Thm. 5.4]. In particular, ρ_{Kahn} is not trivial for biquaternion division algebras. In §3.2.1 (c), we generalise this result.

Chapter 2

Lifting and specialising invariants

"If I have seen farther than others, it is because
I was standing on the shoulders of giants."
— Isaac Newton

In this chapter, we generalise the invariants of §1.4 to central simple k-algebras A with $\mathrm{ind}_k(A)$ possibly not prime to $\mathrm{char}(k)$. We use a lift from positive characteristic to characteristic zero to obtain this as in characteristic zero, the invariants mentioned are always defined. This method is generic, i.e. it does not depend on the precise definition of any of the invariants, but just on the existence. This allows us to perform the lift for a general invariant and then we retrieve the generalisations for any of the invariants mentioned before.

As a warmer-up, we perform such a lift for central simple k-algebras when $\mathrm{char}(k) = p > 0$, but still $p \nmid \mathrm{ind}_k(A)$. In this case the invariants are already defined, but this gives us some techniques and terminology to treat the general case where we perform a similar lift using Kato's logarithmic differentials. The content of this chapter was first treated by the author in [Wou3].

2.1 Moderate case

In this first section, we hence start off by lifting from moderate characteristic to characteristic 0. We explain our strategy (for both the moderate and the wild case). We postpone explicit and detailed arguments to the next (sub)sections.

2.1.1 Strategy

Let k be a field of $\mathrm{char}(k) = p > 0$, let A be a central simple k-algebra with $\mathrm{ind}_k(A) = n \in k^\times$, and let r be any integer. Consider k as a residue field of a ring R which is complete for a discrete valuation v and such that $K = \mathrm{Frac}(R)$ is of characteristic 0. Then A lifts to an Azumaya R-algebra B and $B_K = B \otimes_R K$ is a central simple K-algebra (of same period, degree and index as A), actually the lifted central simple algebra of §1.1 (d). Suppose we are given an invariant $\rho' \in \mathrm{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*_{n,\mathcal{B}_K^{\otimes r}})$. The approach consists of two steps.

(i) *Constructing an auxiliary invariant.* – To construct an invariant $\rho \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$, we first construct an auxiliary invariant $\tilde{\rho} \in \mathrm{Inv}^3(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$. Hence for any field extension k' of k we have to define a morphism

$$\tilde{\rho}_{k'} : \mathbf{SK}_1(A)(k') \to H^3_{n,A^{\otimes r}}(k').$$

So, let K' be a field complete for a discrete valuation w with residue field k' such that K' is a field extension of K and such that w extends v. Due to an isomorphism $\mathbf{SK}_1(B_K)(K') \to \mathbf{SK}_1(A)(k')$ and the residue $H^4_{n,B_K^{\otimes r}}(K') \to H^3_{n,A^{\otimes r}}(k')$, we are able to construct the morphism $\tilde{\rho}_{k'}$.

This morphism is not necessarily an invariant as the functoriality in field extensions is not immediately obtained. There exist after all different possibilities of finding field extensions K' as above. We are able to resolve this aspect using p-rings which are sufficiently canonical.

(ii) *Deducing the required invariant.* – As the residue of cycle modules appears in a functorial short exact sequence (1.6), we obtain an invariant in $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$ as soon as $\tilde{\rho}$ is trivial. By Lemma 1.9, to prove $\tilde{\rho}$ is trivial, it suffices to show that the invariant $\tilde{\pi}(\tilde{\rho})$ of $\mathbf{SL}_1(A)$ is trivial. For that purpose, we use Merkurjev's morphism θ (1.8). So we show $\theta(\tilde{\pi}(\tilde{\rho})) = 0$ carrying out some calculations on $\tilde{A}^0$-groups and using essential results obtained by Kahn and Merkurjev.

We can summarise the strategy by the slogan:

Lift and specialise

By this we mean that in the diagram

$$
\begin{tikzcd}
 & \mathbf{SK}_1(A)(k') \arrow[r, "\cong"] \arrow[d, dotted] \arrow[drr, bend left] & \mathbf{SK}_1(B_K)(K') \arrow[d] & & \\
0 \arrow[r] & H^4_{n,A^{\otimes r}}(k') \arrow[r] & H^4_{n,B_K^{\otimes r}}(K') \arrow[r] & H^3_{n,A^{\otimes r}}(k') \arrow[r] & 0,
\end{tikzcd}
$$

we first construct the bended arrow $\mathbf{SK}_1(A)(k') \to H^3_{n,A^{\otimes r}}(k')$ using a *lift* and the existence of $\rho_K : \mathrm{SK}_1(B_K)(K) \to H^4_{n,B_K^{\otimes r}}(K')$. Then we prove it is zero so that we can *specialise* ρ_K to find the (dotted) invariant of $\mathbf{SK}_1(A)$.

2.1.2 Lifting objects

Before lifting invariants, we have to be able to lift the objects we are working with in a proper way. We explain how to lift fields and central simple algebras.

(a) Central simple algebras – For any field k, we can find a complete discrete valuation ring R such that k is the residue field (e.g. a p-ring R associated with k – see (b)). Denote by K the fraction field of R.

The way of lifting central simple k-algebras to central simple K-algebras is passing by *Azumaya R-algebras* (of constant rank) . These are the natural generalisations of central simple algebras to any ring, see [KO, Ch. III, §§5,6]. They also come with a splitting $A \otimes_R S \cong M_n(S)$ for a faithfully flat R-algebra S and one can also define the Brauer group $\mathrm{Br}(R)$ of R as equivalence classes of Azumaya algebras.

Now let $P(R)$, respectively $P(k)$, be the set of isomorphism classes of Azumaya R-algebras, respectively central simple k-algebras. Then the residue map $P(R) \to P(k)$ associating with the isomorphism class of an Azumaya R-algebra B the class of $B \otimes_R k$, is bijective [Gro2, Thm. 6.1]. So given any central simple k-algebra A, we can find a *lifted Azumaya R-algebra* B of A (i.e. such that $B \otimes_R k \cong A$). Then $B_K = B \otimes_R K$ is a central simple K-algebra of same index and degree as A.

The bijection $P(R) \to P(k)$ induces furthermore an isomorphism $\mathrm{Br}(R) \cong \mathrm{Br}(k)$, and base extension from R to K gives an injection $\mathrm{Br}(R) \to \mathrm{Br}(K)$

[AG, Thm. 7.2]. So in total we have an injection $\mathrm{Br}(k) \to \mathrm{Br}(K)$. Hence B_K has also the same period as A. For an integer $n \in k^\times$, this coincides on the n-torsion part with the injection ${}_n\mathrm{Br}(k) \to {}_n\mathrm{Br}(K)$ from (1.4). This explains why we worked in §1.1 (d) with a lifted central simple algebra with a subscript K.

Remark 2.1 – These morphisms can also be retrieved in a more general way, using the group scheme $\mathbf{PGL}_{R,\infty}$ as $\mathrm{Br}(R) \cong H^1_{\text{ét}}(R, \mathbf{PGL}_{R,\infty})$ - see [KO, Ch. III, Cor. 6.7] and [Mil1, Ch. III, Cor. 4.7 & p.134]. Indeed Grothendieck proves that for any smooth R-group scheme $\mathfrak{G}$ with special fibre $\mathbf{G}$ specialisation gives an isomorphism $H^1_{\text{ét}}(R, \mathfrak{G}) \cong H^1(k, \mathbf{G})$ [SGA, Exp. XXIV, Prop. 8.1]. We refer to this result as *Hensel's lemma à la Grothendieck*. Now $\mathbf{PGL}_{R,\infty}$ is a smooth R-scheme, so we retrieve the isomorphism $\mathrm{Br}(R) \cong \mathrm{Br}(k)$. Furthermore, as $\mathrm{Spec}(K)$ can be considered as an open of $\mathrm{Spec}(R)$, we get from a long exact sequence from étale cohomology $\mathrm{Br}(R) \hookrightarrow H^1(K, \mathbf{PGL}_{K,\infty}) = \mathrm{Br}(K)$ [Mil1, Ch. III, Prop. 1.25].

The power of this lifting of algebras is that $\mathbf{SK}_1(A)(k)$ and $\mathbf{SK}_1(B_K)(K)$ are isomorphic. This result is essentially due to Platonov for central division algebras. The valuation v on K extends to any central division K-algebra D with valuation $w = \frac{1}{m} v \circ \mathrm{Nrd}_{D/K}$ on D where $m > 0$ is the generator of $v \circ \mathrm{Nrd}_{D/K}(D) \subset \mathbb{Z}$ [Ser1, Ch. XII, §2]. Let $\mathcal{O}_D$ be the valuation algebra of w and $\mathcal{P}_D$ its maximal ideal, then we denote by $\overline{D} = \mathcal{O}_D/\mathcal{P}_D$ the *residual division k-algebra* – see also [Wad, §2]. We say that D is *unramified* over K if $[\overline{D} : k] = [D : K]$ and if $Z(\overline{D})$ is separable over k. The residue map $\mathcal{O}_D \to \overline{D}$ restricts to a residue morphism $\mathbf{SL}_1(D)(K) \to \mathbf{SL}_1(\overline{D})(k)$, and Platonov proves the following *rigidity property*.

Theorem 2.2 ([Pla, Prop. 3.4, Thm. 3.12, Cor. 3.13])

Let K be a field complete for a discrete valuation v with residue field k and D an unramified central division K-algebra. The residue morphism

$$\mathbf{SL}_1(D)(K) \to \mathbf{SL}_1(\overline{D})(k)$$

is surjective with kernel contained in $[D^\times, D^\times]$. This induces an isomorphism

$$\mathbf{SK}_1(D)(K) \cong \mathbf{SK}_1(\overline{D})(k).$$

From this we try to deduce an isomorphism between $\mathbf{SK}_1(A)(k)$ and $\mathbf{SK}_1(B_K)(K)$. We use of course Wedderburn's theorem and the Morita invariance of SK_1.

Corollary 2.3

Let A, B, k, R and K as above, then

$$\mathbf{SK}_1(A)(k) \cong \mathbf{SK}_1(B_K)(K).$$

Proof. By Wedderburn's theorem, $B_K \cong M_m(D)$ for a central division K-algebra D and an integer $m > 0$. By the injectivity of $\mathrm{Br}(R) \to \mathrm{Br}(K)$, we find that $M_m(\mathcal{O}_D)$ is Brauer-equivalent to B. So, again by Wedderburn's theorem, $A \cong M_m(\overline{D})$ and it is clear that D is unramified. Hence, Theorem 2.2 and the Morita invariance of SK_1 guarantee that

$$\mathbf{SK}_1(B_K)(K) \cong \mathbf{SK}_1(D)(K) \cong \mathbf{SK}_1(\overline{D})(k) \cong \mathbf{SK}_1(A)(k).$$

■

Remark 2.4 – This isomorphism is also functorial in the following sense. Suppose K' is a field extension of K which is also complete for a discrete valuation v' extending v. Let k' be the residue field of K', which is a field extension of k. Then the isomorphism from above commutes with base extension of K to K' and k to k'. There is of course no equivalence of functors as there is no bijection between field extensions of k and those of K.

(b) p-rings – p-rings provide a sufficiently canonical way of lifting fields of positive characteristic to rings of characteristic zero. Let us start by recalling the definition of these p-rings.

Definition 2.5

A *p-ring* is a complete discrete valuation ring whose residue field is of characteristic $p > 0$ and whose maximal ideal is generated by p.

The name "p-ring" is as in [Mat, §23], but we always suppose them to be complete. This is because in the sequel we only use complete p-rings.

Starting from a field k of $\mathrm{char}(k) = p > 0$, Schoeller gives a explicit construction of p-rings with residue field k [Sch, §3]. They are subrings of the ring of (infinite) Witt vectors over k. Rings of Witt vectors are generalisations of the construction of the p-adic integers $\mathbb{Z}_p$ out of $\mathbb{Z}/p\mathbb{Z}$. See [Wit1, §1] or also [Ser1, Ch.II §6] for more details.

When k is perfect, the p-ring is exactly the ring of Witt vectors over k. In general, the p-ring contains the ring of Witt vectors of the maximal perfect subfield of k. Also note that these p-rings are of mixed characteristic, so they indeed provide a way to perform lifts from positive characteristic to characteristic zero. Let us recall the following important result of these p-rings which allows to perform a lift of invariants.

Theorem 2.6 ([Coh], see also [Gro1, Thm. 19.8.6])

(i) Let W be a p-ring, C a complete local noetherian ring and I an ideal of C not equal to C. Then any local homomorphism $u : W \to C/I$ factors in $W \xrightarrow{v} C \to C/I$, where v is a local homomorphism.

(ii) Let k a field of characteristic $p > 0$. Then there exists a p-ring W with residue field isomorphic to k. If W' is a second p-ring with residue field k', then any isomorphism $u : k \to k'$ descends by quotient from an isomorphism $v : W \to W'$.

Remark 2.7 – Remark that property (i) induces that p-rings are initial objects in the category of complete local noetherian rings with a fixed residue field. This theorem seems to suggest that there exists a universal property of p-rings. However, the induced morphisms do not have to be unique. They are if and only if the residue field k of the p-ring is perfect. So by lack of uniqueness, we call this harmed universal property a *versal* property as Serre does [GMS, §5].

Example 2.8 (of non-uniqueness) – An example of non-uniqueness of the morphism is by the previous remark to be found in non-perfect fields and the most standard example of a non-perfect field gives us easily such examples.

The Laurent series field $\mathbb{F}_p((t))$ is the most common non-perfect field for a prime p. Denote by F is the field consisting of those series $\sum_{i\in\mathbb{Z}} a_i t^i$ with coefficients in $\mathbb{Q}_p$ bounded below for the p-adic valuation and such that $\lim_{i\to-\infty}|a_i|_p = 0$. Then the p-adic valuation v on $\mathbb{Q}_p$ extends to F by defining the valuation of a series as the infimum of the p-adic valuations of its coefficients. The valuation ring $\mathcal{O}_v$ is given by similar series with all coefficients in $\mathbb{Z}_p$. Moreover, $\mathcal{O}_v$ is clearly a p-ring of $\mathbb{F}_p((t))$. (See also [Ras, Ex. 2.3].)

Take an element $u \in \mathbb{Z}_p^\times$ with residue $1 \in \mathbb{F}_p^\times$. Then

$$\mathcal{O}_v \to \mathcal{O}_v \quad \text{defined by} \quad t \mapsto ut$$

is a well defined automorphism and when passing to the residue field $\mathbb{F}_p((t))$ it gives us the identity. Hence the identity map on $\mathbb{F}_p((t))$ induces (infinitely) many choices for lifts to an automorphism of $\mathcal{O}_v$.

Fortunately, on the cohomological level we are not constrained by these scars.

Corollary 2.9

Let W, W' be p-rings such that the residue field k' of W' is a field extension of k, the residue field of W. Denote by $u : k \to k'$ this inclusion. Theorem 2.6 (i) provides a local homomorphism $v : W \to W'$. Let A be a central simple k-algebra with $\mathrm{ind}_k(A) = n \in k^\times$ and lifted Azumaya W-algebra B. Denote furthermore $K = \mathrm{Frac}(W)$ and $K' = \mathrm{Frac}(W')$. Now v defines for any integers $i, n, r \geq 0$ an homomorphism of split exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i+1}_{n,A^{\otimes r}}(k) & \longrightarrow & H^{i+1}_{n,B_K^{\otimes r}}(K) & \xrightarrow{\partial^i} & H^{i}_{n,A^{\otimes r}}(k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle v_*} & & \downarrow{\scriptstyle u_*} & & \\
0 & \longrightarrow & H^{i+1}_{n,A^{\otimes r}}(k') & \longrightarrow & H^{i+1}_{n,B_K^{\otimes r}}(K') & \xrightarrow[\partial^i]{} & H^{i}_{n,A^{\otimes r}}(k') & \longrightarrow & 0.
\end{array}$$

Moreover, v_* does not depend on the choice of v. If $k = k'$, we find in particular an isomorphism $H^{i+1}_{n,B_K^{\otimes r}}(K) \cong H^{i+1}_{n,B_K^{\otimes r}}(K')$.

Proof. The local homomorphism v sends by definition of a morphism the uniformiser $p \in W$ to $p \in W'$. So the diagram and independence of choice of

v follow immediately from the splitting of (1.6) by taking the cup product with the class of p. If u is an isomorphism, v is also an isomorphism by Theorem 2.6 (ii), hence one finds an isomorphism of short exact sequences. ■

To ease the notation and our discussion, we introduce a notion of triples.[1]

Definition 2.10

If F is a (complete) field equipped with a discrete valuation v, then we say $(F, \mathcal{O}_v, \kappa(v))$ is a *(complete) valuation triple* (recall the notations and conventions on page x). A valuation triple (K, R, k) where R is a p-ring (for a prime $p > 0$) is called a *p-triple*. A (finite, resp. separable, resp. Galois) *p-extension* (K', R', k') of (K, R, k) is a p-triple such that k' is a (finite, resp. separable, resp. Galois) field extension of k.

Remark 2.11 – Given a field k of $\operatorname{char}(k) = p > 0$, Theorem 2.6 (ii) gives us a (non-unique) p-triple (K, R, k) *associated with* k. Even more if (K', R', k') is a (finite, resp. separable, resp. Galois) p-extension of (K, R, k), Theorem 2.6 (i) implies that K' is a (finite, resp. unramified, resp. Galois) extension of K – see also [Ser1, §III.5].

If (K, R, k) is a p-triple, F an R-field and $(F, \mathcal{O}_v, \kappa(v))$ a valuation triple such that $\kappa(v)$ is also an R-field, then one says that $(F, \mathcal{O}_v, \kappa(v))$ is an *R-valuation triple*.

Remark 2.12 – We can reformulate the functorial property of the isomorphism of Corollary 2.3 as formulated in Remark 2.4 using p-extensions as follows. For any p-extension (K', R', k') of (K, R, k), we have a commutative diagram:

$$\begin{array}{ccc}
\mathbf{SK}_1(A)(k) & \xrightarrow{\cong} & \mathbf{SK}_1(B_K)(K) \\
\downarrow & & \downarrow \\
\mathbf{SK}_1(A)(k') & \xrightarrow[\cong]{} & \mathbf{SK}_1(B_{K'})(K').
\end{array}$$

[1] Any use of terminology is purely coincidental and has nothing to do with the author's love for craft beer.

The difference in cumbrousness between Remarks 2.4 and 2.12 gives immediately a feeling why it is useful to introduce the notion of triples.

2.1.3 The lift

We have now done the necessary preparations to lift and specialise invariants in moderate characteristic.

Theorem 2.13 — Let k be a field of $\mathrm{char}(k) = p > 0$ and A a central simple k-algebra with $\mathrm{ind}_k(A) = n \in k^\times$. Denote by (K, R, k) a p-triple associated with k, by B the lifted Azumaya R-algebra of A, and let $\rho' \in \mathrm{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*_{n,\mathcal{B}_K^{\otimes r}})$ (for r any integer). There exists a unique $\rho \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$ such that for any p-extension (K', R', k') of (K, R, k) the following diagram commutes

$$\begin{tikzcd} \mathbf{SK}_1(A)(k') \arrow[r, "\rho_{k'}"] & H^4_{n,A^{\otimes r}}(k') \arrow[d] \\ \mathbf{SK}_1(B_K)(K') \arrow[u, "\cong"] \arrow[r, "\rho'_{K'}"'] & H^4_{n,B_K^{\otimes r}}(K'). \end{tikzcd} \tag{2.1}$$

Remark 2.14 – The cycle modules $\mathcal{H}^*_{n,\mathcal{B}_K^{\otimes r}} = (H^j_{n,B_K^{\otimes r}})_{j\geq 2}$ with base K and $\mathcal{H}^*_{n,\mathcal{A}^{\otimes r}} = (H^j_{n,A^{\otimes r}})_{j\geq 2}$ with base k are as described in §1.2 (d). They are the cycle modules obtained by restricting the cycle module $\mathcal{H}^*_{n,\mathcal{B}^{\otimes r}}$ with base R respectively to K and k. Note also that the morphism $H^4_{n,A^{\otimes r}}(k') \to H^4_{n,B_K^{\otimes r}}(K')$ is the injection of the short exact sequence (1.6).

First, we carry out the second step of the general strategy explained in §2.1.1. This relies heavily on the following proposition. We refer to e.g. [Mil3] for the terminology related to algebraic groups.

Proposition 2.15 (Merkurjev [Mer3, Lem. 4.8 and Prop. 4.9])

Let k be a field and $\mathbf{G}$ a semi-simple, simply connected algebraic k-group, then $\tilde{A}^0(\mathbf{G}, \mathcal{H}_n^3) = 0$ for any $n \in k^\times$. In particular (by §1.3 (b)), $\mathrm{Inv}^3(\mathbf{G}, \mathcal{H}_n^*) = 0$.

We allow us to tweak this result by a couple of homological arguments to the following helpful result.

Corollary 2.16

Let k be a field, $\mathbf{G}$ a semi-simple, simply connected algebraic k-group, and A a central simple k-algebra such that $\mathrm{ind}_k(A) = n \in k^\times$, then $\mathrm{Inv}^3(\mathbf{G}, \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}) = 0$ for any integer r.

Remark 2.17 – For $r \equiv 0 \mod \mathrm{per}_k(A)$, we retrieve Proposition 2.15.

Proof. By (1.8), it suffices to prove $\tilde{A}^0(\mathbf{G}, \mathcal{H}^3_{n,\mathcal{A}^{\otimes r}})$ to be trivial. First, we consider the commutative diagram

$$\begin{tikzcd}
H^1(k,\mu_n) \arrow[r] \arrow[d, "\cup r[A]"] & H^1(k(\mathbf{G}),\mu_n) \arrow[r, "\partial^1"] \arrow[d, "\cup r[A_{k(\mathbf{G})}]"] & \bigoplus_{x\in\mathbf{G}^{(1)}} H^0(k(x),\mathbb{Z}/n\mathbb{Z}) \arrow[d, "\oplus_{x\in\mathbf{G}^{(1)}}\cup r[A_{k(x)}]"] \\
H^3_n(k) \arrow[r] \arrow[d] & H^3_n(k(\mathbf{G})) \arrow[r, "\partial^3"] \arrow[d] & \bigoplus_{x\in\mathbf{G}^{(1)}} H^2_n(k(x)) \arrow[d] \\
H^3_{n,A^{\otimes r}}(k) \arrow[r] & H^3_{n,A^{\otimes r}}(k(\mathbf{G})) \arrow[r, "\partial^3_{A^{\otimes r}}"] & \bigoplus_{x\in\mathbf{G}^{(1)}} H^2_{n,A^{\otimes r}}(k(x)),
\end{tikzcd} \tag{2.2}$$

where the rows are chain complexes, the central one being exact by Proposition 2.15. It suffices to show the exactness of the lower row. Kummer theory and the properties of residues [GMS, Rem. 6.2] show that ∂^1, a sum of residues, is actually the principle divisor morphism:

$$k(\mathbf{G})^\times/(k(\mathbf{G})^\times)^n \to \bigoplus_{x\in\mathbf{G}^{(1)}} \mathbb{Z}/n\mathbb{Z} = \mathrm{Div}(\mathbf{G})/n\mathrm{Div}(\mathbf{G}) : \overline{f} \mapsto \overline{\mathrm{div}(f)}.$$

This morphism is however surjective as $\operatorname{Pic}(\mathbf{G}) = 0$ [San, Lem. 6.9].

The exactness of the lower chain complex follows by a diagram chase. Indeed, suppose $x \in H^3_n(k(\mathbf{G}))$ such that $\partial^3_{A^{\otimes r}}(\bar{x}) = 0$ for $\bar{x}$ the image of x in $H^3_{n,A^{\otimes r}}(k(\mathbf{G}))$. Then the surjectivity of ∂^1 gives us $y \in H^1(k(\mathbf{G}), \mu_n)$ such that $x - \big(y \cup [A^{\otimes r}_{k(\mathbf{G})}]\big) \in \ker \partial^3$. The exactness of the middle row gives us then $\bar{x} \in H^3_{n,A^{\otimes r}}(k)$ as required. ■

Proof of Theorem 2.13. Let $\rho' \in \operatorname{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*_{n,\mathcal{B}_K^{\otimes r}})$. We first construct $\tilde{\rho} \in \operatorname{Inv}^3(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$ (as explained in §2.1.1). So we first have to define $\tilde{\rho}_{k'} : \mathbf{SK}_1(A)(k') \to H^3_{n,A^{\otimes r}}(k')$ for any field extension k' of k, and then prove functoriality in field extensions. So, let (K', R', k') be a p-extension of (K, R, k) associated with k'. Then we surely have a morphism $\rho'_{K'} : \mathbf{SK}_1(B_K)(K') \to H^4_{n,B_K^{\otimes r}}(K')$. Denote by π the isomorphism $\mathbf{SK}_1(B_{K'})(K') \to \mathbf{SK}_1(A)(k')$ of Corollary 2.3, then we define

$$\tilde{\rho}_{k'} = \partial^4_{A^{\otimes r}} \circ \rho'_{K'} \circ \pi^{-1} : \mathbf{SK}_1(A)(k') \to H^3_{n,A^{\otimes r}}(k').$$

Remark that this construction does not depend on the particular choice of the p-extension. Indeed, if (K'', R'', k') is another p-extension associated with k', Corollary 2.9 gives an isomorphism of split exact sequences like (1.6) with the identity on the factors $H^4_{n,A^{\otimes r}}(k')$ and $H^3_{n,A^{\otimes r}}(k')$. Moreover, $\partial^4_{A^{\otimes r}}$, $\rho'_{K'}$, and π are functorial for such field extensions, so this constructs indeed an invariant $\tilde{\rho} \in \operatorname{Inv}^3(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$.

Corollary 2.16 and Lemma 1.9 show that $\tilde{\rho} = 0$. So for $a \in \mathbf{SK}_1(A)(k')$, we get that $\rho'_{K'} \circ \pi^{-1}(a)$ comes from a unique element in $H^4_{n,A^{\otimes r}}(k')$ (by the short exact sequence (1.6)). This way, we again get a morphism $\rho_{k'} : \mathbf{SK}_1(A)(k') \to H^4_{n,A^{\otimes r}}(k')$. As before, the short exact sequence (1.6) is functorial and the choice of p-ring has no influence on the definition, so this does define an invariant $\rho \in \operatorname{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$.

The commutative diagram (2.1) follows immediately by the construction, and the uniqueness follows from the injectivity of $H^4_{n,A^{\otimes r}}(k') \to H^4_{n,B_K^{\otimes r}}(K')$ and Corollary 2.16. ■

Remark 2.18 – As the exact sequence (1.6) is split, we could have defined the specialised invariant just using the splitting. This would us not have given us the same diagram we have right now (2.1). Moreover, with our method we are sure not to lose information in degree 3. On the other hand as a result of our method, we do find that the two methods give exactly the same invariant.

Remark 2.19 – For a field k of $\mathrm{char}(k) = p > 0$ and a central simple k-algebra A of $\mathrm{ind}_k(A) \in k^\times$, the invariants from §1.4 are already defined. If (K, R, k) is p-triple, B the lifted Azumaya R-algebra, and ρ any of the invariants $\rho_{\mathrm{S91},B_K}, \rho_{\mathrm{S06},B_K}, \rho_{r,B_K}$ or ρ_{Kahn,B_K}, then it is to be expected that the specialised invariant of ρ is the same as the original one for $\mathbf{SK}_1(A)$.

To obtain this compatibility, one can verify that these invariants verify a lifting property as in Theorem 2.13 (i.e. there is a commutative diagram as (2.1) with ρ the original invariant for $\mathbf{SK}_1(A)$ and ρ' the invariant for $\mathbf{SK}_1(B_K)$). If we refer to these specialised invariants of $\mathbf{SK}_1(A)$, we denote them distinctly by $\tilde{\rho}_{\mathrm{S91},A}, \tilde{\rho}_{\mathrm{S06},A}, \tilde{\rho}_{r,A}$, and $\tilde{\rho}_{\mathrm{Kahn},A}$ to stress the (a priori) difference.

2.2 Wild case

Let k be a field of characteristic $p > 0$ and A a central simple k-algebra with $\mathrm{ind}_k(A) = n$ possibly divisible by p. We enter now a new world, as the cycle module $\mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}$ is not adjusted to our goals. Indeed, as $\mu_{p^n}(k_s)$ is trivial, the Galois cohomology groups $H^{j+1}(k, \mu_{p^n}^{\otimes j})$ are trivial as well. Moreover Kummer's exact sequence (1.1) does not exist any more, so we no longer have an isomorphism of $H^2(k, \mu_{p^n})$ with ${}_{p^n}\mathrm{Br}(k)$ as in the moderate case.

In this section, we describe new cohomology groups (introduced by Kato [Kat1]) which give in this wild case an isomorphism with ${}_{p^n}\mathrm{Br}(k)$. We need such an isomorphism in order to define relative cycle modules as in §1.1 (d). They are furthermore equipped with a short exact sequence comparable to (1.4). This gives us all the ingredients we need to lift and specialise. We carry out this job in the case when the central simple algebra has index p^n. In Section 2.3, we deduce the general case from it using the Brauer decomposition of a central division algebra.

2.2.1 Cohomology groups

In this section, let (K, R, k) be a p-triple and F an R-field. Let us first recall the notion of logarithmic differentials of Kato (ibid.) and the definition of $H^{q+1}_{p^n}(k)$ along with (some of) its properties (for integers $n, q \geq 0$)[2]. Nowadays, the differentials are often defined using *de Rham-Witt complexes*.

(a) Logarithmic differentials – The definition of $H^{q+1}_{p^n}(k)$ is the most explicit for $n = 1$ and this also explains the terminology. So let $\Omega^q_k = \bigwedge \Omega^1_{k/\mathbb{Z}}$ and let $d : \Omega^{q-1}_k \to \Omega^q_k$ be the usual exterior derivative (if $q = 0$, we set $d = 0$). Then, $H^{q+1}_p(k)$ is defined as cokernel of the *Cartier morphism*

$$F - 1 : \Omega^q_k \to \Omega^q_k / d\Omega^{q-1}_k, \quad \text{defined by}$$

$$x\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_q}{y_q} \mapsto (x^p - x)\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_q}{y_q} \quad \text{mod } d\Omega^{q-1}_k,$$

with $x \in k$, $y_1, \ldots, y_q \in k^\times$, and $F(x) = x^p$ [Car, Ch. 2, §6]. The kernel of this morphism is traditionally denoted by $\nu_1(q)_k$.

(b) Generalisation – We can generalise this definition of $H^{q+1}_p(k)$ to a definition of $H^{q+1}_{p^n}(k)$ for any integer $n > 0$ (for $n = 0$, set $H^{q+1}_{p^n}(k) = 0$). This is however quite formal and it is no longer clear why we speak about cohomology of logarithmic differentials. We start from

$$D^q_{p^n}(k) = W_n(k) \otimes \underbrace{k^\times \otimes \ldots \otimes k^\times}_{q \text{ times}},$$

where $W_n(k)$ is the group of p-Witt vectors of length n on k. Now we quotient out by a subgroup generated by the exact relations so that for $n = 1$ we end up with the cohomology of logarithmic differentials under an identification

$$x\frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_q}{y_q} \quad \leftrightarrow \quad x \otimes y_1 \otimes \ldots \otimes y_q, \tag{2.3}$$

for $x \in k$ and $y_1, \ldots, y_q \in k^\times$. So let first $J'_q(k)$ be the subgroup of $D^q_{p^n}(k)$ generated by the elements of the form

(i) $w \otimes b_1 \otimes \ldots \otimes b_q$, satisfying $b_i = b_j$ for $1 \leq i < j \leq q$.

[2] The superscript $q + 1$ is again due to tradition, but is also quite natural in this case.

Then, $C^q_{p^n}(k) = D^q_{p^n}(k)/J'_q(k)$ is a generalisation of logarithmic differentials. Note that the antisymmetry also holds for this generalisation as $w \otimes b_1 b_2 \otimes b_1 b_2 \otimes \dots b_q = 0$ $(w \in W_n(k), b_1, \dots, b_q \in k^\times)$.

Subsequently we introduce cohomology. Note that these groups are equipped with a derivative $d : C^{q-1}_{p^n}(k) \to C^q_{p^n}(k)$: for $a, b_2, \dots, b_q \in k^\times$ and $q > 0$ defined by

$$(0, \dots, 0, a, 0, \dots, 0) \otimes b_2 \otimes \dots \otimes b_q \mapsto (0, \dots, 0, a, 0, \dots, 0) \otimes a \otimes b_2 \otimes \dots \otimes b_q.$$

For $q = 0$, we again set $d = 0$. The cohomology group $H^{q+1}_{p^n}(k)$ is then defined as the cokernel of the Cartier morphism

$$\begin{array}{rcll} F-1: \; C^q_{p^n}(k) & \to & C^q_{p^n}(k)/dC^{q-1}_{p^n}(k), & \text{defined by} \\ w \otimes b_1 \otimes \dots \otimes b_q & \mapsto & (w^{(p)} - w) \otimes b_1 \otimes \dots \otimes b_q. & \end{array}$$

Here, $F(w) = w^{(p)} = (a_1^p, \dots, a_n^p)$ for $w = (a_1, \dots, a_n)$. For $q < 0$, we set $H^{q+1}_{p^n}(k) = 0$. It is clear that this gives us a generalisation under the identification (2.3). In conformity with the case $n = 1$, we denote by $\nu_n(q)_k$ the kernel of the Cartier morphism. Alternatively, $H^{q+1}_{p^n}(k) \cong D^q_{p^n}(k)/J_q(k)$ where $J_q(k)$ is the subgroup of $D^q_{p^n}(k)$ generated by elements of the form (i) and [Kat1, Proof of Thms. 1& 2]

(ii) $(0, \dots, 0, a, 0, \dots, 0) \otimes a \otimes b_2 \otimes \dots \otimes b_q$,

(iii) $(w^{(p)} - w) \otimes b_1 \otimes \dots \otimes b_q$.

Define dlog : $k_s^\times \to \nu_n(1)_{k_s} : a \mapsto (1, 0, \dots, 0) \otimes a$. A calculation with Witt vectors and tensor products gives a short exact sequence of Γ_k-modules: [Car, Ch. 2, Prop. 8]

$$1 \longrightarrow k_s^\times \xrightarrow{p^n} k_s^\times \xrightarrow{\text{dlog}} \nu_n(1)_{k_s} \longrightarrow 1.$$

The associated long exact sequence induces (using Hilbert 90) an isomorphism on the p^n-torsion part of the Brauer group: $H^1(k, \nu_n(1)_{k_s}) \cong {}_{p^n}\mathrm{Br}(k)$. On the other hand, we have an exact sequence

$$0 \longrightarrow \nu_n(q)_{k_s} \longrightarrow C^q_{p^n}(k_s) \xrightarrow{F-1} C^q_{p^n}(k_s)/dC^{q-1}_{p^n}(k_s) \longrightarrow 0. \tag{2.4}$$

The surjectivity of $F-1$ follows from Theorem 2.21 (infra) which proves $H^{q+1}_{p^n}(k_s)=0$ for any $q\geq 0$ and $n>0$. Indeed, if k is the residue field of a field K complete for a discrete valuation, then k_s is the residue field of K_{nr}. As $C^q_{p^n}(k_s)$ is a k_s-vector space such that $C^q_{p^n}(k_s)^{\Gamma_k}=C^q_{p^n}(k)$, we get by the additive version of Hilbert 90 an isomorphism

$$H^1(k,\nu_n(q)_{k_s})\cong H^{q+1}_{p^n}(k). \tag{2.5}$$

So as in the moderate case we find

$$H^2_{p^n}(k)\cong {}_{p^n}\mathrm{Br}(k). \tag{2.6}$$

Remark 2.20 – Comparable to the moderate case (Remark 1.2), the class of a p-algebra $[a,b)_p$ corresponds to $a\,db/b\in H^2_p(k)$ [GS, Prop. 9.2.5].

(c) Kato's exact sequence – As announced, there is also an exact sequence as (1.4). Kato's theory of cohomology of logarithmic differentials is slightly more difficult, but we still have the following result.

Theorem 2.21 (Kato [Kat1], Izhboldin [Izh])

Let $(F,\mathcal{O}_v,\kappa(v))$ be a complete valuation triple and let

$$H^{q+1}_{p^n,\mathrm{nr}}(F)=\ker[H^{q+1}_{p^n}(F)\to H^{q+1}_{p^n}(F_{\mathrm{nr}})].$$

Then we have a split short exact sequence

$$0\to H^{q+1}_{p^n}(\kappa(v))\to H^{q+1}_{p^n,\mathrm{nr}}(F)\to H^q_{p^n}(\kappa(v))\to 0. \tag{2.7}$$

Remark 2.22 – Let us explain the splitting and morphisms without giving proofs. Depending on the characteristics of F and $\kappa(v)$, there are three situations to be discussed.

- In the case of *mixed characteristic* ($\mathrm{char}(F)=0$ and $\mathrm{char}(\kappa(v))=p$), the splitting is obtained by morphisms due to Kato [Kat1, Proof of Thms. 1& 2]. Let first i be the canonical homomorphism

$$\begin{aligned} W_n(\kappa(v))/\{w^{(p)}-w|w\in W_n(\kappa(v))\} &\overset{\varphi}{\cong} H^1(\kappa(v),\mathbb{Z}/p^n\mathbb{Z})\\ &\hookrightarrow H^1(F,\mathbb{Z}/p^n\mathbb{Z}).\end{aligned}$$

The last injection is defined as in the short exact sequence (1.4) and the isomorphism φ comes from the additive version of Hilbert 90 applied to the long exact sequence obtained from Witt's short exact sequence [Wit1, §5]:

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow W_n(\kappa(v)_s) \xrightarrow{x^{(p)}-x} W_n(\kappa(v)_s) \longrightarrow 0.$$

Note that this short exact sequence is actually an instance of (2.4) (for $q = 0$). Then on the one hand we have an inclusion $i^* : H^{q+1}_{p^n}(\kappa(v)) \to H^{q+1}_{p^n,\mathrm{nr}}(F)$ of degree 0, defined by

$$w \otimes \bar{b}_1 \otimes \ldots \otimes \bar{b}_q \mod J_q(\kappa(v)) \mapsto i(w) \cup h^q_{p^n,F}(b_1, \ldots, b_q).$$

On the other hand, we have an inclusion $\psi : H^q_{p^n}(\kappa(v)) \to H^{q+1}_{p^n,\mathrm{nr}}(F)$ of degree 1, defined by

$$w \otimes \bar{b}_2 \otimes \ldots \otimes \bar{b}_q \mod J_{q-1}(\kappa(v)) \mapsto i(w) \cup h^q_{p^n,F}(\pi, b_2, \ldots, b_q).$$

Here $w \in W_n(\kappa(v))$, π is a fixed uniformiser of F, $b_i \in \mathcal{O}_v^\times$, and $h^q_{p^n,F}$ is the Galois symbol (1.3). Kato shows that $i^* \oplus \psi$ gives us the mentioned isomorphism

$$H^{q+1}_{p^n}(\kappa(v)) \oplus H^q_{p^n}(\kappa(v)) \cong H^{q+1}_{p^n,\mathrm{nr}}(F).$$

The morphisms in (2.7) are the obvious morphisms induced by this isomorphism.

- The case of *equicharacteristic* 0 ($\mathrm{char}(F) = \mathrm{char}(\kappa(v)) = 0$) is like the moderate case. Indeed, $H^{q+1}_{p^n,\mathrm{nr}}(F) = H^{q+1}_{p^n}(F)$, as (1.4) gives us

$$H^{q+1}_{p^n}(F_{\mathrm{nr}}) \cong H^{q+1}_{p^n}(\kappa(v)_s) \oplus H^{q+1}_{p^n}(\kappa(v)_s) = 0.$$

- The case of *equicharacteristic* p ($\mathrm{char}(F) = \mathrm{char}(\kappa(v)) = p$) is described by Izhboldin [Izh, Prop. 6.8]. In this case the morphism $i^* : H^{q+1}_{p^n}(\kappa(v)) \to H^{q+1}_{p^n,\mathrm{nr}}(F)$ is defined by

$$\bar{w} \otimes \bar{b}_1 \otimes \ldots \otimes \bar{b}_q \mod J_q(\kappa(v)) \mapsto w \otimes b_1 \otimes \ldots \otimes b_q \mod J_q(F).$$

On the other hand, there is again a morphism $\psi : H^q_{p^n}(\kappa(v)) \to H^{q+1}_{p^n,\mathrm{nr}}(F)$, defined by

$$\bar{w} \otimes \bar{b}_2 \otimes \ldots \otimes \bar{b}_q \mod J_{q-1}(\kappa(v)) \mapsto w \otimes \pi \otimes b_2 \otimes \ldots \otimes b_q \mod J_q(F),$$

where π is again a fixed uniformiser of F, $b_i \in \mathcal{O}_v^\times$, $w = (a_1, \ldots, a_n) \in W_n(\mathcal{O}_v)$, and $\bar{w} = (\bar{a}_1, \ldots, \bar{a}_n)$ its residue in $W_n(\kappa(v))$. Izhboldin shows that $i^* \oplus \psi$ induces a splitting of $H^{q+1}_{p^n,\mathrm{nr}}(F)$. Also in this case, the morphisms in (2.7) are the obvious ones induced by this isomorphism.

(d) Definition of the R-cycle module $\mathcal{H}^*_{p^n,L}$ – Now we can define our cycle module needed to generalise the invariants.

Definition 2.23

Let (K, R, k) be a p-triple with a finite Galois p-extension $(L, S, \overline{L})$. For any integer $n > 0$, we define $\mathcal{H}^*_{p^n,L} = (\mathcal{H}^i_{p^n,L})_{i>0}$ as the cycle module with base R and $\mathcal{H}^{j+1}_{p^n,L}(F) = H^{j+1}_{p^n,L}(F)$ where

$$H^{j+1}_{p^n,L}(F) = \begin{cases} \ker[H^{j+1}_{p^n}(F) \to H^{j+1}_{p^n}(F \otimes_K L)] & \text{if } F \in K\text{-}\mathfrak{fields}, \\ \ker[H^{j+1}_{p^n}(F) \to H^{j+1}_{p^n}(F \otimes_k \overline{L})] & \text{if } F \in k\text{-}\mathfrak{fields}. \end{cases}$$

Remark 2.24 – Note that for any $F \in K$-$\mathfrak{fields}$ the cohomology groups are usual Galois cohomology groups and for $F \in k$-$\mathfrak{fields}$ the cohomology groups are the freshly introduced ones. Remark that $F \otimes_K L$ (or $F \otimes_k \overline{L}$) is not necessarily a field. However as L is finitely separable over K, $F \otimes_K L$ is a finite product of finite separable field extensions of L [Mil4, Thm. 1.18]. Then the cohomology groups can be interpreted as étale cohomology groups (in characteristic zero) or as the finite direct sum of the cohomology groups defined before (in both characteristics).

Remark 2.25 – If $(L_1, S_1, \overline{L}_1)$ and $(L_2, S_2, \overline{L}_2)$ are two finite Galois p-extension of (K, R, k), then there exists a finite Galois p-extension $(L, S, \overline{L})$ of (K, R, k) which is a common p-extension of both $(L_1, S_1, \overline{L}_1)$ and $(L_2, S_2, \overline{L}_2)$. In this case, there exist injections $\mathcal{H}^*_{n,L_1} \to \mathcal{H}^*_{n,L}$ and $\mathcal{H}^*_{n,L_2} \to \mathcal{H}^*_{p^n,L}$. This indicates that the choice of L does not play a big role.

The reason why we need to fix an L at all is in order to obtain a well-defined cycle module with a nice short exact sequence as in (1.4). If we forget about this L, it is not possible to define the residues (D4) in full generality.

Using direct limits of $\mathcal{H}^*_{p^n,L}$'s where $\overline{L}$ runs over all finite Galois extensions of k, we can replace $\overline{L}$ by k_s (and L by K_{nr}). The data and the rules behave well under taking direct limits: the proofs of the analogous statements can always be reduced to the finite case. We leave the adding-in of direct limits as an exercise for the reader who is interested in such a result. In our construction, we do not need to go to the separable closure (see Remark 2.41).

We still have to show that this defines a cycle module. So, we need to define the four data D1-D4 (see §1.2 (a)). The data D1, D2, and D3 only occur in equicharacteristics, while datum D4 can occur in mixed characteristics.

The definition of the functoriality (D1) is straightforward. For a finite extension E of F, we define datum D2. Remark that $E \otimes_F C^q_{p^n}(F) \cong C^q_{p^n}(E)$. One defines a trace on $C^q_{p^n}(E)$ using the trace $\mathrm{Tr}_{E/F}$ of E to F:

$$C^q_{p^n}(E) \cong E \otimes_F C^q_{p^n}(F) \xrightarrow{\mathrm{Tr}_{E/F} \otimes \mathrm{id}} F \otimes_F C^q_{p^n}(F) \cong C^q_{p^n}(F).$$

This extends in a natural way to a definition of D2 on the cohomology groups $H^{q+1}_{p^n,L}(F)$.

(e) $K_m(F)$-module structure (D3) – Take the data as in Definition 2.23. If $\mathrm{char}(F) = 0$ (i.e. F is an extension of K), the $K_m(F)$-module structure is defined as in the moderate case. If $\mathrm{char}(F) = p$ (i.e. F is an extension of k), this structure is inspired by the *differential symbol* in stead of the Galois symbol. For any $m \geq 1$,

$$\rho^m_F : K_m(F) \to \Omega^m_F, \qquad \text{defined by} \qquad \{x_1, \ldots, x_m\} \mapsto \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_m}{x_m},$$

is an homomorphism. Indeed, $d(ab) = bd(a) + ad(b)$ induces $\frac{d(ab)}{ab} = \frac{da}{a} + \frac{db}{b}$, and if $a + b = 1$, we have $\frac{da}{a} \wedge \frac{db}{b} = 0$ as $da + db = 0$ ($a, b \in k^\times$). So ρ^m_F induces a map $K_m(F)/pK_m(F) \to \Omega^m_F$ as $\mathrm{char}(F) = p$ (and so $dx^p = 0$). Even more, the image is clearly contained in $\nu_1(m)_F$. The differential symbol is the morphism

$$h^m_{p,F} : K_m(F)/pK_m(F) \to \nu_1(m)_F.$$

Bloch-Kato-Gabber prove this is actually an isomorphism [BK, Thm. 2.1].

Inspired by this definition, we can propose the following $K_m(F)$-module structure

$$\begin{aligned}\rho^m_{p^n,F} : K_m(F) \times H^{q+1}_{p^n}(K) &\to H^{q+m+1}_{p^n}(F), \quad \text{defined by} \\ (\{x_1, \ldots, x_m\}, w \otimes b_1 \otimes \ldots \otimes b_q) &\mapsto w \otimes x_1 \otimes \ldots \otimes x_m \otimes b_1 \otimes \ldots \otimes b_q.\end{aligned}$$

The same arguments as above guarantee this is well defined. For $a \in K_m(F)$ and $b \in H^{q+1}_{p^n}(F)$, we denote the scalar multiplication by $a \cdot b = \rho^m_{p^n,F}(a, b)$. This structure restricts to a $K_m(F)$-module structure on $(H^{q+1}_{p^n,L}(F))_{q\geq 0}$ for $(L, S, \overline{L})$ as in Definition 2.23. Indeed if $b \in J_q(F \otimes \overline{L})$, we have $a \cdot b \in J_{q+m}(F \otimes \overline{L})$ for any $a \in K_m(F)$.

(f) The residue and an exact sequence – We are left with the task to define a residue (datum D4), and we also would like to generalise the short exact sequence (1.4).

Proposition 2.26

Let (K, R, k) be a p-triple and $(L, S, \overline{L})$ a finite Galois p-extension. For any complete R-valuation triple $(F, \mathcal{O}_v, \kappa(v))$ and for all integers $n > 0$ and $q \geq 0$, we have a split short exact sequence:

$$0 \to H^{q+1}_{p^n,L}(\kappa(v)) \to H^{q+1}_{p^n,L}(F) \to H^{q}_{p^n,L}(\kappa(v)) \to 0. \tag{2.8}$$

Proof. We certainly have two versions of the sequence (2.7):

$$\begin{tikzcd} 0 \arrow[r] & H^{q+1}_{p^n}(\kappa(v)) \arrow[r] \arrow[d] & H^{q+1}_{p^n,\mathrm{nr}}(F) \arrow[r] \arrow[d] & H^{q}_{p^n}(\kappa(v)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{q+1}_{p^n}(\kappa(v) \otimes \overline{L}) \arrow[r] & H^{q+1}_{p^n,\mathrm{nr}}(F \otimes L) \arrow[r] & H^{q}_{p^n}(\kappa(v) \otimes \overline{L}) \arrow[r] & 0. \end{tikzcd}$$

As the sequences are split and the splittings respect the commutative diagram, the split exact sequence follows from the snake lemma. Here, $H^{q+1}_{p^n,\mathrm{nr}}(F \otimes L)$ is to be interpreted in the same way as in Remark 2.24. ∎

Remark 2.27 – The residues for an R-field F complete for a discrete valuation v are defined by this sequence. Suppose F is endowed with a discrete valuation, but is not complete for the topology defined by this valuation. Then take a completion $\hat{F}$ of F with respect to v. The residue field of $\hat{F}$ is then equal to the residue field $\kappa(v)$ of F, and in this case the residue is defined (in the same way as in §1.1 (c)) as composition of

$$H^{i+1}_{p^n,L}(F) \to H^{i+1}_{p^n,L}(\hat{F}) \to H^{i}_{p^n,L}(\kappa(v)).$$

Hence, we have introduced the four required data to have a cycle module along with this practical short exact sequence. One also has to verify all the rules of the cycle modules. We refer to Appendix A for a detailed computation. The only non-trivial rule is actually C and this follows from the rule C for the Milnor K-groups using the Bloch-Kato isomorphism and the Bloch-Kato-Gabber isomorphism.

(g) Relative version – As in §1.1 (d), we define relative cycle modules using isomorphism (2.6) and the action of K-theory – similar to the alternative definition (1.5) of the moderate cycle module.

Definition 2.28

Let (K, R, k) be a p-triple, A a central simple k-algebra of $\mathrm{ind}_k(A) = p^n$, and B the lifted Azumaya R-algebra. Let $(L, S, \overline{L})$ be a finite Galois extension of (K, R, k) such that $\overline{L}$ is a splitting field of A. We define for any integer r a cycle module $\mathcal{H}^*_{p^n,L,\mathcal{B}^{\otimes r}} = (\mathcal{H}^j_{p^n,L,\mathcal{B}^{\otimes r}})_{j\geq 2}$ with base R by

$$\mathcal{H}^{j+1}_{p^n,L,\mathcal{B}^{\otimes r}}(F) = H^{j+1}_{p^n,L,B^{\otimes r}}(F) = H^{j+1}_{p^n,L}(F)/(K_{j-1}(F)\cdot r[B_F]),$$

with $F \in R$-$\mathfrak{fields}$ and $[B_F]$ be the class of $B_F = B \otimes_R F$ in ${}_{p^n}\mathrm{Br}(F)$.

Remark 2.29 – Note that $B_F = A_F$ if F is a field extension of k. In this case we also use the notation $H^{j+1}_{p^n,L,A^{\otimes r}}(F)$. For a field extension F of K, we also use the notation $H^{j+1}_{p^n,L,B_K^{\otimes r}}(F)$. If we restrict $\mathcal{H}^*_{p^n,L,\mathcal{B}^{\otimes r}}$ to field extensions of k (resp. K) as in §1.2 (b), we write it similarly as $\mathcal{H}^*_{p^n,L,\mathcal{A}^{\otimes r}}$ (resp. $\mathcal{H}^*_{p^n,L,\mathcal{B}_K^{\otimes r}}$). Note that for $r \equiv 0 \mod \mathrm{per}_k(A)$, we find $H^{j+1}_{p^n,L,\mathcal{B}^{\otimes r}}(F) = H^{j+1}_{p^n,L}(F)$ (cfr. Remark 1.5).

Remark 2.30 – The choice of $\overline{L}$ is possible by (a more enhanced version of) Wedderburn's theorem which gives us a finite separable extension L' of k splitting A. We obtain $\overline{L}$ by taking a finite extension of L' such that $\overline{L}/k$ is Galois. Then we associate a p-triple $(L, S, \overline{L})$ with $\overline{L}$.

We can even suppose $\overline{L}$ to be a cyclic extension of k. Indeed, Albert's theorem [Alb2, Thm. 18] states that any central simple k-algebra of degree p^n is Brauer-equivalent to a cyclic k-algebra (as in Example I.5).

The fact that we choose $\overline{L}$ to be a splitting field of A is to guarantee that the scalar multiplication ends up in $\mathcal{H}^*_{p^n,L}$. Indeed, for an extension F of k, the base extension morphism $\mathrm{Br}(F) \to \mathrm{Br}(F \otimes \overline{L})$ sends the class of $[A_F]$ to zero, and hence the same holds for the subgroup $K_{j-1}(F) \cdot r[A_F]$. Also for a field extension F of K, the subgroup $K_{j-1}(F) \cdot r[B_F]$ is trivial after base extension by L. This follows from the previous statement and §2.1.2 (a).

We still have to verify that this relative definition gives us indeed a cycle module. We base ourselves of course on the fact that the absolute one is a cycle module and we verify that the data are well defined modulo the subgroups taken into account.

Data D1, D2, and D3 follow more or less immediately from the definition as the fields appearing in these data have the same characteristic. Datum D4 for equicharacteristics also follows from the definition of the residue of $\mathcal{H}^*_{p^n,L}$. So it suffices to verify datum D4 for the case of mixed characteristic. In addition, we have to generalise the exact sequence (2.8). As D4 is defined using this exact sequence, it even suffices just to generalise the exact sequence (2.8).

Proposition 2.31

Using the same notations as in Definition 2.28, we have for any R-valuation triple $(F, \mathcal{O}_v, \kappa(v))$ a split short exact sequence

$$0 \to H^{q+1}_{p^n,L,B^{\otimes r}}(\kappa(v)) \to H^{q+1}_{p^n,L,B^{\otimes r}}(F) \to H^{q}_{p^n,L,B^{\otimes r}}(\kappa(v)) \to 0. \quad (2.9)$$

Proof. By the previous remarks, it suffices to prove the proposition in the case of mixed characteristic. The goal is to verify that (2.8) commutes with inclusions in a commutative diagram (for $q \geq 2$ and up to a sign):

$$\begin{tikzcd}
0 \arrow[r] & H^{q+1}_{p^n,L}(\kappa(v)) \arrow[r, "i^*"] & H^{q+1}_{p^n,L}(F) \arrow[r, "\partial"] & H^{q}_{p^n,L}(\kappa(v)) \arrow[r] & 0 \\
0 \arrow[r] & K_{q-1}(\kappa(v)) \cdot r[A_{\kappa(v)}] \arrow[u, hook] \arrow[r, dotted] & K_{q-1}(F) \cdot r[B_F] \arrow[u, hook] \arrow[r, dotted] & K_{q-2}(\kappa(v)) \cdot r[A_{\kappa(v)}] \arrow[u, hook] \arrow[r] & 0.
\end{tikzcd}$$

Let us first verify that the diagram

$$\begin{tikzcd}
H^2_{p^n}(\kappa(v)) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^2_{p^n,\mathrm{nr}}(F) \arrow[d, "\cong"] \\
{}_{p^n}\mathrm{Br}(\kappa(v)) \arrow[r, "i"'] & {}_{p^n}\mathrm{Br}_{\mathrm{nr}}(F)
\end{tikzcd} \tag{2.10}$$

commutes, where $\mathrm{Br}_{\mathrm{nr}}(F) = \ker(\mathrm{Br}(F) \to \mathrm{Br}(F_{\mathrm{nr}}))$, i^* is the morphism of the short exact sequence (2.7), and i is the injection of §2.1.2 (a). The verification is a straightforward computation with cocycles. Let us carry this out. First, take a generator $a \otimes \bar{x} \in H^2_{p^n}(\kappa(v))$ with $a \in W_n(\kappa(v))$ and $x \in \mathcal{O}_v^\times$. Then,

$$i^*(a \otimes \bar{x}) = \left((\tau(y)/y)^{\bar{\sigma}(b)-b}\right)_{\sigma,\tau} \in H^2_{p^n}(F)$$

with $y^p = x$ and $a = b^p - b$ for well chosen $y \in F_{\mathrm{nr}}^\times$ and $b \in W_n(F_{\mathrm{nr}})$. Here we consider $\bar{\sigma}(b) - b$ as an element of $\mathbb{Z}/p^n\mathbb{Z}$ (with $\bar{\sigma}$ the residue of $\sigma \in \Gamma_F$ in $\Gamma_{\kappa(v)}$). Then the image in ${}_{p^n}H^2(F, F_s^\times) \cong {}_{p^n}\mathrm{Br}(F)$ is represented by the same expression. On the other hand, the image of $a \otimes \bar{x} \in H^2_{p^n}(\kappa(v))$ in ${}_{p^n}H^2(\kappa(v), \kappa(v)_s^\times) \cong {}_{p^n}\mathrm{Br}(\kappa(v))$ is $c = \left((\sigma(\bar{y})/\bar{y})^{\tau(\bar{b})-\bar{b}}\right)_{\sigma,\tau}$. So,

$$i(c) = \left((\sigma(y)/y)^{\bar{\tau}(b)-b}\right)_{\sigma,\tau} \in H^2_{p^n}(F).$$

As i^* is defined by a cup product, this equals $-i^*(a \otimes \bar{x})$.

The restriction of (2.10) to the subgroups gives a commutative diagram (up to a sign)

$$\begin{tikzcd} H^2_{p^n,L}(\kappa(v)) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^2_{p^n,L}(F) \arrow[d, "\cong"] \\ {}_{p^n}\mathrm{Br}(\overline{L} \otimes_k \kappa(v)/\kappa(v)) \arrow[r, "i"'] & {}_{p^n}\mathrm{Br}(L \otimes_K F/F). \end{tikzcd}$$

The proof of this proposition hence follows immediately from this fact as i^*, ∂, and ψ (see Remark 2.22) respect the K-theory module structure and as the sign disappears when taking quotients. So,

$$\begin{aligned} i^*\big(K_{q-1}(\kappa(v)) \cdot r[A_{\kappa(v)}]\big) &= i^*_K\big(K_{q-1}(\kappa(v))\big) \cdot i^*\big(r[A_{\kappa(v)}]\big) \\ &\subset K_{q-1}(F) \cdot r[B_F], \\ \partial\big(K_{q-1}(F) \cdot r[B_F]\big) &= \partial_K\big(K_{q-1}(F)\big) \cdot r[A_{\kappa(v)}] \\ &= K_{q-2}(\kappa(v)) \cdot r[A_{\kappa(v)}], \qquad \text{and} \\ \psi\big(K_{q-2}(\kappa(v)) \cdot r[A_{\kappa(v)}]\big) &= \psi_K\big(K_{q-2}(\kappa(v))\big) \cdot i^*\big(r[A_{\kappa(v)}]\big) \\ &\subset K_{q-1}(F) \cdot r[B_F]. \end{aligned}$$

Here i^*_K, ∂_K, and ψ_K are maps in Milnor's K-theory defined as in §1.2 (d). ■

Remark that this exact sequence also satisfies a property as Corollary 2.9, as also in this case the splittings are given by a choice of uniformiser (see Remark 2.22) which is canonical for p-rings.

Corollary 2.32

Take the notations of Definition 2.28 and let (K', R', k') be a p-extension of (K, R, k). Denote by $u : k \to k'$ the inclusion. Theorem 2.6 (i) gives a local homomorphism $v : R \to R'$ which defines for any integers $i, n \geq 0$ an homomorphism of split exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{i+1}_{p^n,L,A^{\otimes r}}(k) & \longrightarrow & H^{i+1}_{p^n,L,B_K^{\otimes r}}(K) & \xrightarrow{\partial^i} & H^{i}_{p^n,L,A^{\otimes r}}(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle v_*} & & \downarrow{\scriptstyle u_*} & & \\ 0 & \longrightarrow & H^{i+1}_{p^n,L,A^{\otimes r}}(k') & \longrightarrow & H^{i+1}_{p^n,L,B_K^{\otimes r}}(K') & \xrightarrow{\partial^i} & H^{i}_{p^n,L,A^{\otimes r}}(k') & \longrightarrow & 0. \end{array}$$

Moreover, v_* does not depend on the choice of v. If $k = k'$, we find in particular an isomorphism $H^{i+1}_{p^n,L,B_K^{\otimes r}}(K) \cong H^{i+1}_{p^n,L,B_K^{\otimes r}}(K')$.

2.2.2 The lift

Before lifting, we prove a result analogous to the one of Merkurjev (Proposition 2.15). This is an immediate consequence of a result of Kahn which uses Zariski cohomology groups and reduced Zariski cohomology groups:

$$\tilde{H}^0_{Zar}(\mathbf{G}, \mathcal{H}^3_{p^n}) \cong H^0_{Zar}(\mathbf{G}, \mathcal{H}^3_{p^n})/H^3_{p^n}(k).$$

Here, $\mathcal{H}^3_{p^n}$ is the functor $k\text{-}\mathfrak{fields} \to \mathfrak{Ab}$ associated with the cohomology of logarithmic differentials (see also §3.2.2). This uses also notions about algebraic groups, we refer to e.g. [Mil3] for the definitions.

Theorem 2.33 (Kahn [Kah1])

Let k be a field of $\mathrm{char}(k) = p > 0$, $\mathbf{G}$ a semi-simple, simply connected, absolutely almost simple algebraic k-group, $\overline{\mathbf{G}} = \mathbf{G} \times_k k_s$, and $n > 0$ an integer. If $\mathrm{CH}^2(\mathbf{G}) = 0$, then the base extension $\mathbf{G} \to \overline{\mathbf{G}}$ induces an injection

$$\tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n}) \hookrightarrow H^0_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{H}^3_{p^n}).$$

Remark 2.34 – The proof consists of putting together various results. The author apologises for the non-transparency of the arguments and the plenty of references to the literature, but he hopes it improves the readability of the whole of this passage. For further details on the objects mentioned in both the theorem and the proof, the reader can find more information in the references. These are only used as auxiliary objects and therefore they are not explained in full details.

Proof. Let $\Gamma = \Gamma_k$ be the absolute Galois group of k. Using motivic cohomology à la Lichtenbaum, Kahn constructs a morphism (ibid., first complex after the diagram p. 406)

$$\tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n}) \to \mathbb{H}^5(\overline{\mathbf{G}}/k_s, \Gamma(2))^\Gamma \tag{2.11}$$

with kernel contained in $H^1(F, H^1_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{K}_2))$. Here, $\mathbb{H}^5(\mathbf{G}/k_s, \Gamma(2))$ is an hypercohomology group defined by Kahn as the (fifth) étale hypercohomology of a relative complex based on the Lichtenbaum complex $\Gamma(2)$ [Lic], and $\mathcal{K}_2$ is the Zariski sheaf obtained from the presheaf $U \mapsto K_2^Q(U)$ (where K_2^Q is Quillen's K-theory). In order to define this morphism, $H^0_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{K}_2) \cong K_2^Q(k_s)$ has to hold; this is due to Esnault-Kahn-Levine-Viehweg [EKLV, Prop. 3.20 (i)]. As $H^1_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{K}_2) \cong \mathbb{Z}$ [Gil1, Prop. 1'], the morphism (2.11) is injective (see [Kah1, diagram p. 406.]). Using $\mathrm{CH}^2(\overline{\mathbf{G}})^\Gamma = 0$ [EKLV, Prop. 3.20 (iii)] and the following injection of Kahn (ibid., exact sequence (18) p. 404), we find a desired injective morphism:

$$\mathbb{H}^5(\overline{\mathbf{G}}/k_s, \Gamma(2))^\Gamma \hookrightarrow H^0_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{H}^3_{p^n}).$$

It follows from the computations in [Kah1] that this morphism is indeed the natural map induced by base extension. ∎

Corollary 2.35

Let k be a field of characteristic $p > 0$, L a finite Galois extension of k, and $\mathbf{G}$ a semi-simple, simply connected, absolutely almost simple algebraic k-group such that $\mathrm{CH}^2(\mathbf{G}) = 0$. Then $\mathrm{Inv}^3(\mathbf{G}, \mathcal{H}^*_{p^n,L}) = 0$ for any integer $n > 0$.

Remark 2.36 – Here, $\mathcal{H}^3_{p^n,L}$ is the cycle module of Definition 2.23 restricted to k-$\mathfrak{fields}$ as in §1.2 (b). To ease notation, we use L in stead of $\overline{L}$ which appears in Definition 2.23.

Proof. By (1.8) it suffices to show that $\tilde{A}^0(\mathbf{G}, \mathcal{H}^3_{p^n,L}) = 0$. As Rost proves $A^i(\mathbf{G}, M_j) \cong H^i_{\mathrm{Zar}}(\mathbf{G}, M_j)$ for a cycle module M and integers i, j [Ros2, Cor. 6.5], it suffices to show that $\tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n,L}) = 0$. So let $x \in \tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n,L}) \subset \tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n})$. We know that $H^3_{p^n}(k(\mathbf{G})) \to H^3_{p^n}(k_s(\mathbf{G}))$ factors through $H^3_{p^n}(k(\mathbf{G}) \otimes L)$. So $x \in \ker\left[H^3_{p^n}(k(\mathbf{G})) \to H^3_{p^n}(k_s(\mathbf{G}))\right]$ as $x \in H^3_{p^n,L}(k(\mathbf{G}))$, and hence $x \in \ker[\tilde{H}^0_{\mathrm{Zar}}(\mathbf{G}, \mathcal{H}^3_{p^n}) \to H^0_{\mathrm{Zar}}(\overline{\mathbf{G}}, \mathcal{H}^3_{p^n})]$. Theorem 2.33 gives $x = 0$. ■

The arguments used in the proof of Theorem 2.13 are purely homological and can be recycled in this wild case if one replaces Proposition 2.15 by Corollary 2.35. Hence we get the following theorem.

Theorem 2.37

Let k be a field of $\mathrm{char}(k) = p > 0$, A a central simple k-algebra of $\mathrm{ind}_k(A) = p^n$, and $\overline{L}$ a finite Galois extension of k that splits A. Let (K, R, k) be a p-triple associated with k and $(L, S, \overline{L})$ a p-triple associated with $\overline{L}$. Let B be the lifted Azumaya R-algebra and $\rho' \in \mathrm{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*_{p^n,L,\mathcal{B}_K^{\otimes r}})$ (for r any integer). There exists a unique $\rho \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{p^n,L,\mathcal{A}^{\otimes r}})$ such that for any p-extension (K', R', k') of (K, R, k) the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{SK}_1(A)(k') & \xrightarrow{\rho_{k'}} & H^4_{p^n,L,A^{\otimes r}}(k') \\
\uparrow \cong & & \downarrow \\
\mathbf{SK}_1(B_K)(K') & \xrightarrow[\rho'_{K'}]{} & H^4_{p^n,L,B_K^{\otimes r}}(K').
\end{array}$$

Remark 2.38 – Recall that the cycle modules $\mathcal{H}^*_{p^n,L,\mathcal{B}_K^{\otimes r}} = (\mathcal{H}^j_{p^n,L,\mathcal{B}_K^{\otimes r}})_{j\geq 2}$ with base K and $\mathcal{H}^*_{p^n,L,\mathcal{A}^{\otimes r}} = (\mathcal{H}^j_{p^n,L,\mathcal{A}^{\otimes r}})_{j\geq 2}$ with base k are the respective restrictions of $\mathcal{H}^*_{p^n,L,\mathcal{B}^{\otimes r}}$ with base R to K and to k (Remark 2.29).

Proof. To generalise the proof of Theorem 2.13, one has to generalise Corollary 2.16. So it suffices to define a diagram as (2.2) since the other arguments are a diagram chase transferable to this wild setting. So let $\mathbf{G} = \mathbf{SL}_1(A)$. We consider the following diagram with exact columns:

$$\begin{tikzcd}
k^\times \arrow[r] \arrow[d, "\cdot r[A]"] & k(\mathbf{G})^\times \arrow[r, "\partial^1"] \arrow[d, "\cdot r[A_{k(\mathbf{G})}]"] & \bigoplus_{x\in\mathbf{G}^{(1)}} \mathbb{Z} \arrow[d, "\oplus_{x\in\mathbf{G}^{(1)}} \cdot r[A_{k(x)}]"] \\
H^3_{p^n}(k) \arrow[r] \arrow[d] & H^3_{p^n}(k(\mathbf{G})) \arrow[r, "\partial^3"] \arrow[d] & \bigoplus_{x\in\mathbf{G}^{(1)}} H^2_{p^n}(k(x)) \arrow[d] \\
H^3_{p^n,A^{\otimes r}}(k) \arrow[r] & H^3_{p^n,A^{\otimes r}}(k(\mathbf{G})) \arrow[r, "\partial^3_{A^{\otimes r}}"] & \bigoplus_{x\in\mathbf{G}^{(1)}} H^2_{p^n,A^{\otimes r}}(k(x)),
\end{tikzcd}$$

Note that $\mathrm{CH}^2(\mathbf{G}) = 0$ as $\mathbf{G}$ is an interior form of $\mathbf{SL}_m(k)$ with $m = \deg_k(A)$ [Pan] and hence the central row in the diagram is exact by Corollary 2.35. Again, ∂^1 is the divisor morphism and as $\mathrm{Pic}(\mathbf{G}) = 0$ [San, Lem. 6.9], ∂^1 is surjective. So, the same diagram chase and a similar construction as in the moderate case finish the proof. ■

We can now deduce generalisations of the invariants of §1.4.

Corollary 2.39

Under the same conditions as in Theorem 2.37, the invariants $\rho_{\mathrm{S91},B_K}, \rho_{\mathrm{S06},B_K}, \rho_{r,B_K}$, and ρ_{Kahn,B_K} induce unique invariants of $\mathbf{SK}_1(A)$ satisfying the lifting property. We denote them respectively by $\tilde{\rho}_{\mathrm{S91},A}, \tilde{\rho}_{\mathrm{S06},A}, \tilde{\rho}_{r,A}$, and $\tilde{\rho}_{\mathrm{Kahn},A}$, and call them the respective *generalised invariants*.

Proof. We have to show that if ρ is any of the given invariants for $\mathbf{SK}_1(B_K)$, then it has values in $\mathcal{H}^4_{p^n,L,\mathcal{B}_K^{\otimes r}}$ (for r the appropriate integer). This immediately follows from the commutative diagram

$$\begin{tikzcd} \mathrm{SK}_1(B_K) \arrow[r, "\rho_K"] \arrow[d] & H^4_{p^n,L,B_K^{\otimes r}}(K) \arrow[d] \\ \mathrm{SK}_1(B_L) \arrow[r, "\rho_L"'] & H^4_{p^n,L,B_K^{\otimes r}}(L) \end{tikzcd}$$

and the triviality of $\mathrm{SK}_1(B_L)$ (as L splits B_K). ■

Remark 2.40 – Note that $\tilde{\rho}_{\mathrm{Kahn},A}$ and $\tilde{\rho}_{\mathrm{S06},A}$ are injective if A is a biquaternion algebra (over a field of even characteristic). Indeed, this follows from the construction and the injectivity of the moderate invariants for biquaternion algebras (see (1.11,1.13) and Remark 1.11).

Remark 2.41 – The definition of these generalised invariants does not depend on the choice of L as in any case the invariants are trivial after base extension to a splitting field of the central simple algebra. In the same way as in Remark 2.25 we could however replace L by k_s.

2.3 General case

We conclude the lifting and specialising procedure by considering the general case. So let k be a field of characteristic $p > 0$ and A a central simple k-algebra of arbitrary index $e = p^n m$ ($p \nmid m$). Wedderburn's theorem gives a unique (up to isomorphism) central division k-algebra D Brauer-equivalent to A. Brauer's decomposition theorem gives central division k-algebras D_{p^n} and D_m of $\mathrm{ind}_k(D_{p^n}) = p^n$ and $\mathrm{ind}_k(D_m) = m$ such that $D \cong D_{p^n} \otimes D_m$. This gives us an isomorphism of functors by (I.4)

$$\mathbf{SK}_1(A) \cong \mathbf{SK}_1(D) \cong \mathbf{SK}_1(D_{p^n}) \oplus \mathbf{SK}_1(D_m).$$

Let us also use a slight abuse of notation and set $A_{p^n} = D_{p^n}$ and $A_m = D_m$.

In order to define the invariants in full generality, we glue the moderate case (Theorem 2.13) and the wild case (Theorem 2.37) together with this isomorphism of $\mathbf{SK}_1(A)$. So we also have to glue to cycle modules together in the obvious way.

Definition 2.42

Let (K, R, k) be a p-triple, A a central simple k-algebra of $\mathrm{ind}_k(A) = e = p^n m$ $(p \nmid m)$, and B the lifted Azumaya R-algebra. Let $\overline{L}$ be a finite Galois extension of k such that it is a splitting field of A_{p^n} and let $(L, S, \overline{L})$ be an associated p-triple. We define for any integer r the following cycle module with base R:

$$\mathcal{H}^*_{e,L,\mathcal{B}^{\otimes r}} = \mathcal{H}^*_{m,\mathcal{B}_m^{\otimes r}} \oplus \mathcal{H}^*_{p^n,L,\mathcal{B}_{p^n}^{\otimes r}}.$$

Here, $\mathcal{B}_m$ and $\mathcal{B}_{p^n}$ correspond to the Brauer decomposition of A (and B_K), we use this notation to keep up with the definitions in §§1.2 (d) and 2.2.1 (g). Using Theorems 2.13 and 2.37, we get the following theorem.

Theorem 2.43

Let k be a field of $\mathrm{char}(k) = p > 0$, A a central simple k-algebra of $\mathrm{ind}_k(A) = e = p^n m$ $(p \nmid m)$, and $\overline{L}$ a finite Galois extension of k splitting A_{p^n}. Let (K, R, k) a p-triple associated with k and $(L, S, \overline{L})$ a p-triple associated with $\overline{L}$. Let B the lifted Azumaya R-algebra and $\rho' \in \mathrm{Inv}^4\left(\mathbf{SK}_1(B_K), \mathcal{H}^*_{e,L,\mathcal{B}_K^{\otimes r}}\right)$ (for r any integer). There exists a unique $\rho \in \mathrm{Inv}^4\left(\mathbf{SK}_1(A), \mathcal{H}^*_{e,L,\mathcal{A}^{\otimes r}}\right)$ such that for any p-extension (K', R', k') of (K, R, k) the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{SK}_1(A)(k') & \xrightarrow{\rho_{k'}} & H^4_{e,L,A^{\otimes r}}(k') \\
\uparrow \cong & & \downarrow \\
\mathbf{SK}_1(B_K)(K') & \xrightarrow[\rho'_{K'}]{} & H^4_{e,L,B_K^{\otimes r}}(K').
\end{array}$$

Then we can define the generalised invariants of $\mathbf{SK}_1(A)$.

Corollary 2.44

Under the same conditions as in Theorem 2.43, the invariants $\rho_{\mathrm{S91},B_K}, \rho_{\mathrm{S06},B_K}, \rho_{r,B_K}$, and ρ_{Kahn,B_K} induce unique invariants of $\mathbf{SK}_1(A)$ satisfying the lifting property. We denote them respectively by $\tilde{\rho}_{\mathrm{S91},A}, \tilde{\rho}_{\mathrm{S06},A}, \tilde{\rho}_{r,A}$, and $\tilde{\rho}_{\mathrm{Kahn},A}$; we call them the respective *generalised invariants.*

2.4 Some remarks

Let us finish this chapter by giving some remarks on our construction.

2.4.1 Other possible constructions

There are a couple of points where we could tweak the construction to obtain actually the same invariant. We did not mention (all of) them at the relevant points, in order to stay focused on our aims at that time. Over here we collect them together.

- As mentioned in §1.2 (b), we could have worked with two different cycle modules. This would be just a matter of notation and noting that there are residue maps from the one cycle module (in characteristic zero) to the other (in positive characteristic)..
- As mentioned in Remark 2.18, we could have used the splitting of the exact sequences (1.6) and (2.9). This a priori gives another diagram of compatibility of invariants. From method presented, it follows however that both constructions give the same invariant.
- In stead of splitting up the discussion into the moderate (prime-to-p) and wild case (p-primary), we could treat them together as Kahn's results to prove Theorem 2.33 also hold in the moderate case. If we would have done this, we had to split up some of the other constructions and proofs into a moderate and a wild case. It seems more structured to split up the discussion at an earlier level.

We can also refine the morphism of Lemma 1.9 to an isomorphism of interest. To do so, we need the following definition.

Definition 2.45

Let k be a field, let $A : k\text{-}\mathfrak{fields} \to \mathfrak{Groups}$ be a group functor, and let M be a cycle module with base k. An invariant $\rho \in \mathrm{Inv}^j(A, M)$ is called *unramified* if for any field extension F of k the composition

$$A\big(F((t))\big) \xrightarrow{\rho} M_j\big(F((t))\big) \xrightarrow{\partial_j} M_{j-1}(F)$$

is trivial. The subgroup of unramified invariants is denoted by $\mathrm{Inv}^j_{\mathrm{nr}}(A, M)$.

Remark 2.46 – Usually unramified objects are defined being trivial passing to any discrete valuation field and not just to a field of Laurent series [CT, Thm. 4.1.1]. This definition also gives us Proposition 2.47, but not immediately Corollary 2.48.

Lemma 1.9 can be proved to restrict to an isomorphism.

Proposition 2.47

Let k be a field, A a central simple k-algebra of $e = \mathrm{ind}_k(A)$, and L a finite Galois splitting field of A. The canonical projection $\mathbf{SL}_1(A) \to \mathbf{SK}_1(A)$ induces an isomorphism for any integers $r, j \geq 0$:

$$\mathrm{Inv}^j_{\mathrm{nr}}(\mathbf{SK}_1(A), \mathcal{H}^*_{e,L,\mathcal{A}^{\otimes r}}) \xrightarrow{\sim} \mathrm{Inv}^j_{\mathrm{nr}}(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,\mathcal{A}^{\otimes r}}).$$

Proof. It is clear that the injection from Lemma 1.9 restricts well to an injection on the unramified subgroups. Hence it remains to prove the surjectivity, so take any $\rho \in \mathrm{Inv}^j_{\mathrm{nr}}(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,\mathcal{A}^{\otimes r}})$. Let k' be any field extension of k, then we prove that $\rho([a,b]) = 0$ for any commutator $[a,b]$ of $A^\times_{k'}$. Set $\alpha(t) = [t + (1-t)a, b]$, a commutator of $A^\times_{k'((t))}$. As ρ is unramified, $\partial^j(\rho(\alpha(t))) = 0$. Glue now the short exact sequences (1.6) and (2.9) into

$$0 \to H^j_{e,L,A^{\otimes r}}(k') \to H^j_{e,L,A^{\otimes r}}(k'((t))) \to H^{j-1}_{e,L,A^{\otimes r}}(k') \to 0.$$

We find that $\rho(\alpha(t))$ is an element of $H^j_{e,L,A^{\otimes r}}(k')$, so it is constant. That gives us

$$0 = \rho(\alpha(0)) = \rho(\alpha(1)) = \rho([a,b]).$$

■

Corollary 2.48

With the same conditions as in Proposition 2.47, we have an isomorphism

$$\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{e,L,A^{\otimes r}}) \xrightarrow{\sim} \mathrm{Inv}^4(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,A^{\otimes r}}).$$

Proof. In view of Lemma 1.9 and Proposition 2.47, it suffices to prove

$$\mathrm{Inv}^4_{\mathrm{nr}}(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,A^{\otimes r}}) \cong \mathrm{Inv}^4(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,A^{\otimes}r}).$$

This follows immediately from Corollary 2.16 and its wild analogue proved in the proof of Theorem 2.37. Indeed, if $\rho \in \mathrm{Inv}^4(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,A^{\otimes r}})$, then

$$\mathbf{SK}_1(A)(F) \to H^4_{e,L,A^{\otimes r}}(F) \to H^4_{e,L,A^{\otimes r}}(F((t))) \to H^3_{e,L,A^{\otimes r}}(F)$$

for F a field extension of k, gives an invariant in $\mathrm{Inv}^3(\mathbf{SL}_1(A), \mathcal{H}^*_{e,L,A^{\otimes r}}) = 0$ ■

In stead of using the injectivity in the construction, we can actually just concentrate on generalising invariants of $\mathbf{SL}_1(A)$ and use Merkurjev's description (1.8). Indeed, by this corollary this amounts to defining invariants of $\mathbf{SK}_1(A)$. To incorporate this immediately in §§2.1, 2.2, 2.3, one first had to prove Corollary 2.16 and its wild analogue (proof of Theorem 2.37). This would have taken about the same effort as now.

2.4.2 Other view point

Using the groups $A^i, \tilde{A}^0$, and A^0_{mult} of §1.2 (c) and §1.3 (b), there is yet another way of looking at the construction. Let (K, R, k) be a p-triple, A a central simple k-algebra of $\mathrm{ind}_k(A) = n$, B the lifted Azumaya R-algebra, $(L, S, \overline{L})$ a finite Galois p-extension of (K, R, k) such that $\overline{L}$ splits A, and $\mathcal{H}^* = \mathcal{H}^*_{n,L,\mathcal{B}^{\otimes r}}$ the cycle module with base R of Definition 2.42 (for r any integer).

Denote $\boldsymbol{\mathcal{G}} = \mathbf{SL}_1(B)$. It is defined like $\mathbf{SL}_1(B_K)$ as the kernel of a reduced norm on B induced by a splitting $B \otimes_R S \cong M_m(S)$ – see [Knu, Ch. III, §1] for more details. The generic fibre $\boldsymbol{\mathcal{G}}_K = \mathbf{SL}_1(B_K)$ is an open of $\boldsymbol{\mathcal{G}}$. Call Z the complement, the image of the special fibre $\mathbf{G} = \mathbf{SL}_1(A)$ in $\boldsymbol{\mathcal{G}}$ under the immersion of schemes $\psi : \mathbf{G} \to \boldsymbol{\mathcal{G}}$. For any integer $i \geq 0$, the points of Z of codimension $i+1$ correspond under ψ to points of codimension i in $\mathbf{G}$. In the same way, Spec(K) is an open of Spec(R) with complement the image of Spec(k). Rost's localising sequence [Ros2, §5] gives exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A^0(R, \mathcal{H}^4) \arrow[r] \arrow[d] & A^0(K, \mathcal{H}^4) \arrow[r] \arrow[d] & A^0(k, \mathcal{H}^3) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^0(\boldsymbol{\mathcal{G}}, \mathcal{H}^4) \arrow[r] & A^0(\boldsymbol{\mathcal{G}}_K, \mathcal{H}^4) \arrow[r] & A^0(\mathbf{G}, \mathcal{H}^3) \arrow[r] & \cdots
\end{tikzcd} \tag{2.12}$$

Corollaries 2.16 and 2.35 (generalised to $\mathcal{H}^*$ in the proof of Theorem 2.37) show that $\tilde{A}^0(\mathbf{G}, \mathcal{H}^3)$ is trivial. Using diagram (2.12), the snake lemma gives an isomorphism

$$\tilde{A}^0(\boldsymbol{\mathcal{G}}_K, \mathcal{H}^4) \cong \tilde{A}^0(\boldsymbol{\mathcal{G}}, \mathcal{H}^4)$$

preserving multiplicative elements. Due to Merkurjev's description (§1.3 (b)), we get an isomorphism

$$\mathrm{Inv}^4(\boldsymbol{\mathcal{G}}_K, \mathcal{H}^*) \cong \tilde{A}^0(\boldsymbol{\mathcal{G}}, \mathcal{H}^4)_{\mathrm{mult}}.$$

The group on the right hand side is defined in the same way as was done for algebraic groups in §1.3 (b). As $\mathcal{H}^*$ has base R, the morphism of schemes $\mathbf{G} \to \boldsymbol{\mathcal{G}}$ gives also a morphism

$$A^0(\boldsymbol{\mathcal{G}}, \mathcal{H}^4) \to A^0(\mathbf{G}, \mathcal{H}^4),$$

giving in the same way a morphism:

$$\tilde{A}^0(\boldsymbol{\mathcal{G}}, \mathcal{H}^4)_{\mathrm{mult}} \to \mathrm{Inv}^4(\mathbf{G}, \mathcal{H}^*). \tag{2.13}$$

In total we obtain a diagram,

$$\begin{tikzcd}
\mathrm{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*) \arrow[r, hook, "\pi"] \arrow[d, dotted] & \mathrm{Inv}^4(\boldsymbol{\mathcal{G}}_K, \mathcal{H}^*) \arrow[d, "\varphi"] \\
\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*) \arrow[r, hook] & \mathrm{Inv}^4(\mathbf{G}, \mathcal{H}^*),
\end{tikzcd}$$

which induces the existence of the dotted arrow. Indeed, let $\rho \in \mathrm{Inv}^4(\mathbf{SK}_1(B_K), \mathcal{H}^*)$ and $(F, S, \overline{F})$ a p-extension of (K, R, k), then $(\varphi \circ \pi(\rho))_{\overline{F}}$ sends commutators of $A_{\overline{F}}^{\times}$ to 0 as they correspond to commutators of $B_F^{\times}$ due to the isomorphism $\mathbf{SK}_1(A)(\overline{F}) \cong \mathbf{SK}_1(B_K)(F)$ (Corollary 2.3).

In Theorem 2.43, we constructed this same dotted arrow by a more explicit construction.

Chapter 3

Comparing invariants

"Ich habe Angst, dass die Mathematik vor dem Ende des Jahrhunderts zugrunde geht, wenn dem Trend nach sinnloser Abstraktion - die Theorie der leeren Menge, wie ich es nenne - nicht Einhalt geboten wird."
— Carl Ludwig Siegel

It is generally assumed that all defined invariants of $\mathbf{SK}_1$ are essentially the same, but very few results exist on this subject. In this chapter, we compare some of the different existing invariants.

First of all, we treat the biquaternion case (Section 3.1). In the Book of Involutions [KMRT, §17], Knus-Merkurjev-Rost-Tignol construct an explicit cohomological invariant ρ_{BI} of $\mathbf{SK}_1(A)$ when A is a biquaternion algebra over k, we call it *KMRT's invariant*. They do not put any restriction on the index. If $\mathrm{char}(k) \neq 2$, they prove their invariant is essentially the same as Suslin's invariant ρ_{S06}. Using the construction of Chapter 2, we prove that for base fields of characteristic 2 ρ_{BI} essentially equals $\tilde{\rho}_{\mathrm{S06}}$.

In Section 3.2 we compare several of the invariants with Kahn's invariant ρ_{Kahn}. Using the fact that ρ_{S91} is non-trivial for Platonov's examples of non-trivial SK_1, we also find that ρ_{Kahn} is not trivial for these examples. We also prove a formula for the value on the centre of the product of two symbol algebras under Kahn's invariant which generalises a formula of Merkurjev for biquaternion algebras.

The results obtained in this chapter were first studied by the author in [Wou2].

3.1 Invariants for biquaternion algebras

The aim of this section is to compare ρ_{BI} in the characteristic 2 case to $\tilde{\rho}_{\mathrm{S06}}$. We first recall the definition of ρ_{BI} which needs Witt groups and Witt rings and also recall why these invariants are essentially the same when the characteristic of the base field is different from 2. Then we are able to do the comparison in the wild case, proving ρ_{BI} satisfies a lifting property.

3.1.1 An explicit invariant

We start by giving the concrete definition of KMRT's invariant. This needs the notion of involutions on Azumaya algebras and Witt groups and rings.

(a) Involutions on Azumaya algebras – In order to define the invariant, a symplectic involution σ on the biquaternion algebra is used. We recall the definition of a symplectic involution on an Azumaya algebra (so in particular on a central simple algebra). We treat this in this general setting of Azumaya algebras, because we need this for our purposes later on. We refer to [Knu, Ch. III, §8] for more details on involutions on Azumaya algebras.

Definition 3.1

Let R be a ring and A an Azumaya algebra over R with an R-linear involution σ. Suppose $\alpha : A \otimes_R S \xrightarrow{\sim} M_n(S)$ is a faithfully flat splitting of A. Then $\tilde{\sigma} = \alpha(\sigma \otimes 1)\alpha^{-1}$ is an involution on $M_n(S)$. Since $x \mapsto \tilde{\sigma}(x^t)$ is an automorphism of $M_n(S)$, we can choose $u \in \mathrm{GL}_n(S)$ such that $\tilde{\sigma}(x) = ux^t u^{-1}$ for all $x \in M_n(S)$. Because $\tilde{\sigma}^2 = 1$, we get $u^t = \epsilon u$ for $\epsilon \in \mu_2(S)$. Then ϵ is called the type of σ (it is well defined and independent of the choice of faithfully flat splitting [Knu, Ch. III, 8.1.1.]). If $2 \neq 0$ in R, an involution of type 1 is called *orthogonal* and an involution of type -1 is called *symplectic*. If $2 = 0$ in R, an involution is called *symplectic* if u as above can be written as $v - v^t$ for $v \in M_n(S)$, otherwise it is called *orthogonal*.

Remark 3.2 – If R is an integral domain, then an involution on an Azumaya algebra can only have type 1 or -1. When k is a field, a central simple k-algebra of odd degree can only have orthogonal involutions, while a central simple algebra of even degree can have involutions of both types [KMRT, Cor. 2.8].

If A is a central simple algebra over k of degree $2n$ with a symplectic involution σ, we can refine the definition of reduced norm, trace, and characteristic polynomial. Set first $\mathrm{Symd}(A,\sigma) = \{a + \sigma(a) \mid a \in A\}$, the vector space of *symmetrised elements* of A under σ. If $a \in \mathrm{Symd}(A,\sigma)$, the reduced characteristic polynomial $\mathrm{Prd}_{a/k}(X)$ is a square [KMRT, Prop. 2.9]. Take $\mathrm{Prp}_{\sigma,a/k}(X)$ the unique monic polynomial such that $\mathrm{Prd}_{a/k}(X) = (\mathrm{Prp}_{\sigma,a/k}(X))^2$; this is the *Pfaffian characteristic polynomial*. The *Pfaffian trace* $\mathrm{Trp}_{\sigma/k}(a)$ and the *Pfaffian norm* $\mathrm{Nrp}_{\sigma/k}(a)$ are defined as coefficients of $\mathrm{Prp}_{\sigma,a/k}(X)$, compatible with the expression of $\mathrm{Nrd}_{A/k}(a)$ and $\mathrm{Trd}_{A/k}(a)$ as coefficients of $\mathrm{Prd}_{a/k}(X)$ (I.1):

$$\mathrm{Prp}_{\sigma,a/k}(X) = X^n - \mathrm{Trp}_{\sigma/k}(a)X^{n-1} + \ldots + (-1)^n \,\mathrm{Nrp}_{\sigma/k}(a).$$

So $\mathrm{Nrd}_{A/k}(a) = (\mathrm{Nrp}_{\sigma/k}(a))^2$ and $\mathrm{Trd}_{A/k}(a) = 2\,\mathrm{Trp}_{\sigma/k}(a)$. For any field extension k' of k, we abbreviate $\mathrm{Prp}_{\sigma_{k'},a'/k'}(X)$ by $\mathrm{Prp}_{\sigma,a'/k'}(X)$ for $a' \in A_{k'}$ and $\sigma'_k = \sigma \otimes_k \mathrm{id}$ the base extension of σ to k' which is a symplectic involution on $A_{k'} = A \otimes_k k'$. Likewise, we use the notation $\mathrm{Trp}_{\sigma/k'}(a')$ and $\mathrm{Nrp}_{\sigma/k'}(a')$ for $a' \in A_{k'}$.

(b) Witt groups – To explain the value group of KMRT's invariant, we need Witt groups and Witt rings.[1] The *Witt group* $W_q(k)$ is the group of Witt-equivalence classes of non-singular quadratic spaces over k with addition defined by the orthogonal sum $\perp$:

- Given two quadratic spaces (V,q) and (V',q') over k, the orthogonal sum $(V,q) \perp (V',q')$ is given by $(V \oplus V', q \perp q')$, where $q \perp q'$ is defined by
$$(q \perp q')(v,v') = q(v) + q(v') \quad (v \in V, v' \in V').$$
- The Witt group $W_q(k)$ consists of non-singular quadratic spaces over k up to *Witt-equivalence*. Two non-singular quadratic spaces (V,q) and (V',q') are Witt-equivalent if $(V,q) \perp M$ is isometric to $(V',q') \perp M'$ for M and M' some *hyperbolic quadratic spaces*. An *hyperbolic plane* is given by $\mathbb{H} = (k^2, [0,0])$, where $[0,0]$ stands for $k^2 \to k : (x,y) \mapsto xy$. An hyperbolic quadratic space is the orthogonal sum of hyperbolic planes.

The *Witt ring* $W(k)$ is the ring of Witt-equivalence classes of non-singular symmetric bilinear spaces with addition given by the orthogonal sum $\perp$ and multiplication by the tensor product $\otimes$:

[1] Do not mix up the Witt group and Witt ring with $W_n(k)$ consisting of the Witt vectors on a field k - see §§2.1.2 (b) and 2.2.1 (b).

- Given two bilinear spaces (V, B) and (V', B') over k, the orthogonal sum $(V, B) \perp (V', B')$ is given by $(V \oplus V', B \perp B')$, where $B \perp B'$ is defined by

  $$(B \perp B')((v, v'), (w, w')) = B(v, w) + B(v', w') \ (v, w \in V, v', w' \in V').$$

  The tensor product $(V, B) \otimes (V', B')$ is given by $(V \otimes V', B \otimes B')$, where $B \otimes B'$ is defined by

  $$(B \otimes B')((v \otimes v'), (w \otimes w')) = B(v, w) \cdot B(v', w') \quad (v, w \in V, v', w' \in V').$$

- The Witt ring $W(k)$ has as elements the non-singular symmetric bilinear spaces over k up to *Witt-equivalence*. Two non-singular bilinear spaces (V, B) and (V', B') are Witt-equivalent if $(V, B) \perp M$ is isometric to $(V', B') \perp M'$ for M and M' *metabolic bilinear spaces*. A *metabolic plane* is given by $\mathbb{H} = (k^2, < a : 1 : 0 >)$, where $a \in k$ and $< a : 1 : 0 >$ stands for the bilinear form B on k^2 with $B(e_1, e_1) = a$, $B(e_2, e_2) = 0$, and $B(e_1, e_2) = 1$ where $\{e_1, e_2\}$ is a k-vector space basis for k^2. A metabolic bilinear space is an orthogonal sum of metabolic planes.

Remark 3.3 – If $\mathrm{char}(k) \neq 2$, we know that as groups (with the orthogonal sum) $W_q(k)$ and $W(k)$ are isomorphic. We are however interested in the characteristic 2 case, so we have to make a clear distinction. For more information on Witt groups and Witt rings in this general case, we refer to [Bae, Ch. I] and [Kah2, Ch. 1] (including the discussion on the characteristic 2 case by Laghribi in [Kah2, App. D]).

Example 3.4 – Suppose that (V, q) is a non-singular quadratic space over k (of $\mathrm{char}(k) \neq 2$) and that $\{e_1, \ldots, e_n\}$ is a orthogonal basis for V (with respect to q). For any $x = \sum_{i=1}^{n} x_i e_i \in V$, we have $q(x) = a_1 x_1^2 + \ldots + a_n x_n^2$ with $a_i = q(e_i) \in k^\times$. Then we denote $(V, q) = \langle a_1, \ldots a_n \rangle$. An *n-fold Pfister form* is given by

$$\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \ldots \otimes \langle 1, -a_n \rangle,$$

for $a_1, \ldots, a_n \in k^\times$. The tensor product of the quadratic forms is induced by the tensor product of the corresponding bilinear forms. These Pfister forms can be generalised in characteristic 2 in a similar way. See (ibid., D.11.2).

We can equip $W_q(k)$ with a $W(k)$-module structure. If (V, B) is a non-singular symmetric bilinear space on k and if (V', q) is a non-singular quadratic space on k, then $(V \otimes V', B \otimes q)$ is a quadratic space on k with $B \otimes q$ defined by

$$(B \otimes q)(v \otimes v') = B(v, v)q(v') \quad \text{for } v \in V, v' \in V'.$$

Let $I(k)$ be the *fundamental ideal* of $W(k)$ (generated by the non-singular bilinear spaces of even dimension). For any integer $n \geq 0$, we set $I^n(k) = (I(k))^n$ (with $I^0(k) = W(k)$) and $I^n W_q(k) = I^n(k) \otimes W_q(k)$. This clearly defines a filtration

$$W_q(k) = I^0 W_q(k) \supset I^1 W_q(k) \supset I^2 W_q(k) \supset \dots$$

We denote the graded quotients by $\overline{I^n W_q(k)} = I^n W_q(k)/I^{n+1} W_q(k)$.

Remark 3.5 – Set $W'_q(k)$ the subgroup of $W_q(k)$ consisting of equivalence classes of even-dimensional non-singular quadratic spaces over k and $I^n W'_q(k) = I^n(k) \otimes W'_q(k)$. Also denote $\overline{I^n W'_q(k)} = I^n W'_q(k)/I^{n+1} W'_q(k)$. If $\mathrm{char}(k) \neq 2$, we have $I^n W'_q(k) = I^{n+1}(k)$ by the equivalence of symmetric bilinear and quadratic spaces. Again, in general we are not able to use this fact.

(c) Definition – Suppose A is a biquaternion algebra over k with a symplectic involution σ. Knus-Merkurjev-Rost-Tignol construct an explicit map [KMRT, Def. 17.5]

$$\mathrm{SL}_1(A) \to \overline{I^3 W'_q(k)} : a \mapsto \begin{cases} 0 & \text{if } \sigma \text{ hyperbolic,} \\ \Phi_v + I^4 W'_q(k) & \text{if } \sigma \text{ not hyperbolic.} \end{cases}$$

with kernel equal to $[A^\times, A^\times]$. Recall that an involution is called *hyperbolic* if there exists an idempotent $e \in A$ such that $\sigma(e) = 1 - e$. Furthermore, Φ_v is the quadratic form

$$A \to k : x \mapsto \Phi_v(x) = \mathrm{Trp}_\sigma(\sigma(x)vx),$$

where $v \in \mathrm{Symd}(A, \sigma) \cap A^\times$ satisfies $v(\mathrm{Trp}_\sigma(v) - v)^{-1} = -\sigma(a)a$. There always exists a v satisfying this condition (ibid., Lem. 17.3). This definition is well defined and independent of the choice of v and σ. Moreover the construction is functorial so that we have an invariant

$$\rho_{\mathrm{BI},A} : \mathbf{SK}_1(A) \to \overline{\mathcal{I}^3 \mathcal{W}'_q},$$

where $\overline{\mathcal{I}^3\mathcal{W}'_q}$ is the functor

$$k\text{-}\mathfrak{fields} \to \mathfrak{Ab} : F \mapsto \overline{I^3W'_q(F)}.$$

Remark 3.6 – The element $v \in \mathrm{Symd}(A,\sigma) \cap A^\times$ in the definition above can be given more explicitly. If $\sigma(a)a = 1$, one can take for v any unit in $\{x \in \mathrm{Symd}(A,\sigma) \mid \mathrm{Trp}_{\sigma/k}(x) = 0\}$. If $\sigma(a)a \neq 1$, the element v is unique and equal to $1 - \sigma(a)a$ (ibid., Lem. 17.3).

3.1.2 Comparison KMRT-Suslin, moderate case

In this section we recall why $\rho_{\mathrm{BI},A}$ and $\rho_{\mathrm{S06},A}$ are equal if A is a biquaternion algebra over k of $\mathrm{char}(k) \neq 2$. This is because both Suslin and Knus-Merkurjev-Rost-Tignol prove their invariant equals $\rho_{\mathrm{Rost},A}$. We already recalled the commutative diagram (1.13) giving the equality of $\rho_{\mathrm{S06},A}$ and $\rho_{\mathrm{Rost},A}$.

To compare ρ_{BI} to ρ_{Rost} famous isomorphisms are used, most of them recently proved. Indeed, there are isomorphisms $\psi^1_F : K_4(F)/2 \to \overline{I^4(F)} = I^4(F)/I^5(F)$ for any F of $\mathrm{char}(F) \neq 2$ (Milnor's conjecture for quadratic forms [Mil5, Q. 4.3], proved by Orlov-Vishik-Voevodsky [OVV, Thm 4.1]) and $\psi^2_F : H^4(F,\mu_2) \to K_4(F)/2$ (Milnor's conjecture [Mil5, §6] or a special case of the Bloch-Kato isomorphism).

So the obvious way of comparing ρ_{BI} and ρ_{Rost} is by the composed isomorphism $\psi_F = \psi^1_F \circ \psi^2_F$. Indeed, Knus-Merkurjev-Rost-Tignol prove that the following diagram commutes [KMRT, Notes §17]:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{SK}_1(A)(F) \arrow[r, "\rho_{\mathrm{Rost},A,F}"] \arrow[d, "="'] & H^4_2(F) \arrow[r] \arrow[d, "\psi"] & H^4_2(F(Y)) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathbf{SK}_1(A)(F) \arrow[r, "\rho_{\mathrm{BI},A,F}"'] & \overline{I^4(F)} \arrow[r] & \overline{I^4(F(Y))},
\end{tikzcd} \tag{3.1}$$

for F any field extension of k and Y the Albert form defined by (1.12).

So combining diagrams (1.13) and (3.1), it follows that ρ_{S06} and ρ_{BI} are the same for biquaternion algebras in characteristic different from 2.

3.1.3 Lifting algebras with involution

We first explain how to lift central simple algebras with a symplectic involution. We do this for general central simple algebras and later on use the result for biquaternion algebras.

(a) Lifting generally – Let (K, R, k) be a p-triple and A an Azumaya algebra over R of degree $2n$ with symplectic involution σ. Define the R-group scheme $\mathbf{PGSp}(A, \sigma) = \mathbf{Aut}(A, \sigma)$, defined for any R-algebra S by

$$\mathbf{Aut}(A, \sigma)(S) = \mathrm{Aut}(A_S, \sigma_S) = \{\varphi \in \mathrm{Aut}_S(A_S) \,|\, \varphi \circ \sigma_S = \sigma_S \circ \varphi\},$$

with $\sigma_S = \sigma \otimes \mathrm{id}$ the canonical extension of σ to $A_S = A \otimes_R S$. It is known that all Azumaya algebras of degree $2n$ with symplectic involutions up to isomorphism are classified by $H^1_{\text{ét}}(R, \mathbf{PGSp}(A, \sigma))$ [KMRT, 29.22]. Since $\mathbf{PGSp}(A, \sigma)$ is a smooth group scheme (proof as in the field case - ibid., p. 347), we can use Hensel's lemma à la Grothendieck to get an isomorphism:

$$H^1_{\text{ét}}(R, \mathbf{PGSp}(A, \sigma)) \cong H^1(k, \mathbf{PGSp}(\overline{A}, \bar{\sigma})),$$

where $\overline{A} = A \otimes_R k$ is the reduced central simple k-algebra and $\bar{\sigma} = \sigma \otimes \mathrm{id}$ is the reduced involution on $\overline{A}$, which is also symplectic. On the other hand, we have an inclusion [Mil1, Ch. III, Prop. 1.25]

$$H^1_{\text{ét}}(R, \mathbf{PGSp}(A, \sigma)) \hookrightarrow H^1(K, \mathbf{PGSp}(A_K, \sigma_K)).$$

So in total we have an inclusion:

$$H^1(k, \mathbf{PGSp}(\overline{A}, \bar{\sigma})) \hookrightarrow H^1(K, \mathbf{PGSp}(A_K, \sigma_K)).$$

Remark 3.7 – Note that this lift coincides with lifting central simple algebras as explained in §2.1.2 (a). Over there, we actually used the same arguments for the smooth R-group scheme $\mathbf{PGL}_{R,\infty}$ (see Remark 2.1).

So starting with a central simple k-algebra A with symplectic involution σ, we find a lifted Azumaya algebra B over R with symplectic involution τ and hence a central simple K-algebra B_K with symplectic involution τ_K. In particular, $\deg_k(A) = \deg_K(B_K)$ and $\mathrm{per}_k(A) = \mathrm{per}_K(B_K)$. Since biquaternion algebras are exactly the central simple algebras of degree 4 and period 1 or 2, we see that a biquaternion algebra over k with a symplectic involution lifts to a biquaternion algebra with a symplectic involution over K.

(b) Lifting explicitly – We can also perform this lift more explicitly in the wild case.[2] The lift in the moderate case is canonical, symbol algebras lift to symbol algebras by lifting the relations. This follows also from Remark 1.2 and the injection defined by (1.4). The wild case is a little bit more complicated. Please be aware of an abuse of notation: both in positive characteristic and in characteristic zero variables u and v are used.

Let (K, R, k) be a 2-triple, $A = [\bar{a}, \bar{b}) \otimes_k [\bar{c}, \bar{d})$ a biquaternion k-algebra where $a, c \in R$ and $b, d \in R^\times$. Then the lifted Azumaya R-algebra is $B = [a, b) \otimes_R [c, d)$ where e.g. $[a, b)$ is the R-algebra generated by u, v satisfying slightly different relations than usual: $u^2+u = a$, $v^2 = b$, and $uv = -v(u+1)$. We can rewrite it as $B = (4a + 1, b)_R \otimes_R (4c + 1, d)_R$, where $(4a + 1, b)_R$ is the R-algebra generated by i, j with $i^2 = 4a + 1$, $j^2 = b$, and $ij = -ji$. Indeed, an isomorphism is given by $i = 2u + 1$ and $j = v$.

- For a symplectic involution on A, it suffices by [KMRT, Prop. 2.23 (1)] to take the product of an orthogonal involution σ_1 on $[\bar{a}, \bar{b})$ and a symplectic involution σ_2 on $[\bar{c}, \bar{d})$. Let σ_1 be defined by $\sigma_1(u) = u, \sigma_1(v) = v$ (and hence $\sigma_1(uv) = uv + v$) and σ_2 defined by $\sigma_2(u) = u + 1, \sigma_2(v) = v$ (and hence $\sigma_2(uv) = uv$). By (ibid., Prop. 2.6 (2)) an involution on a quaternion algebra in characteristic 2 is symplectic if and only if 1 is a symmetrised element. So σ_1 is indeed orthogonal and σ_2 is symplectic as
$$\mathrm{Symd}\Big([\bar{a}, \bar{b}), \sigma_1\Big) = \langle v\rangle \quad \text{and} \quad \mathrm{Symd}\Big([\bar{c}, \bar{d}), \sigma_2\Big) = \langle 1\rangle.$$
So $\sigma = \sigma_1 \otimes \sigma_2$ is a symplectic involution on A. In total we get
$$\mathrm{Symd}(A, \sigma) = \langle 1 \otimes 1, u \otimes 1, v \otimes 1, uv \otimes 1 + v \otimes u, v \otimes v, v \otimes uv\rangle.$$
- To find a lifted symplectic involution on B_K, again by (ibid., Prop. 2.23 (1)) it suffices to take the product of an orthogonal involution τ_1 on $(4a + 1, b)$ and a symplectic τ_2 involution on $(4c + 1, d)$. We try to find these involutions such that τ_1 (resp. τ_2) is a lift of σ_1 (resp. σ_2).

  We see immediately that a lift τ_1 from σ_1 should satisfy $\tau_1(i) = i$ (as $\tau_1(2i + 1) = 2i + 1$), $\tau_1(j) = \pm j$, and hence $\tau_1(ij) = \mp ij$. So, we get two possible lifts: τ_1 defined by $\tau_1(i) = i$, $\tau_1(j) = j$, and $\tau_1(ij) = -ij$, and τ_1' defined by $\tau_1'(i) = i$, $\tau_1'(j) = -j$, and $\tau_1'(ij) = ij$. Then
$$\begin{aligned}\mathrm{Symd}\left((4a + 1, b), \tau_1\right) &= \langle 1, i, j\rangle \qquad \text{and}\\ \mathrm{Symd}\left((4a + 1, b), \tau_1'\right) &= \langle 1, i, ij\rangle.\end{aligned}$$

[2] This calculation is the result of a discussion with Jean-Pierre Tignol.

For a symplectic involution on a quaternion algebra in characteristic different from 2, the vector space of symmetrised elements has dimension 1, while for an orthogonal involution it is of dimension 3 (ibid., Prop. 2.6 (1)). So we see that both τ_1 and τ_1' are orthogonal.

On the other hand, a lift τ_2 from σ_2 should clearly satisfy $\tau_2(i) = -i$, $\tau_2(j) = \pm j$, and hence $\tau_2(ij) = \pm ij$. So we get again two possible lifts: τ_2 defined by $\tau_2(i) = -i$, $\tau_2(j) = -j$, and $\tau_2(ij) = -ij$, and τ_2' defined by $\tau_2'(i) = -i$, $\tau_2'(j) = j$, and $\tau_2'(ij) = ij$. So

$$\begin{aligned} \mathrm{Symd}\left((4c+1, d), \tau_2\right) &= \langle 1 \rangle \qquad \text{and} \\ \mathrm{Symd}\left((4c+1, d), \tau_2'\right) &= \langle 1, j, ij \rangle. \end{aligned}$$

Then τ_2 is a symplectic involution and τ_2' is an orthogonal involution. So we get two possible lifted symplectic involutions on B_K, namely $\tau = \tau_1 \otimes \tau_2$ and $\tau' = \tau_1' \otimes \tau_2$. (If we would have started from another symplectic involution on A, we would have got yet different symplectic involutions on B_K.)

We have

$$\begin{aligned} \mathrm{Symd}(B_K, \tau) &= \langle 1\otimes 1, i\otimes 1, j\otimes 1, ij\otimes i, ij\otimes j, ij\otimes ij\rangle \qquad \text{and} \\ \mathrm{Symd}(B_K, \tau') &= \langle 1\otimes 1, i\otimes 1, ij\otimes 1, j\otimes i, j\otimes j, j\otimes ij\rangle. \end{aligned}$$

Furthermore it follows that

$$\mathrm{Symd}(B, \tau) \otimes_R k = \mathrm{Symd}(A, \sigma) = \mathrm{Symd}(B, \tau') \otimes_R k$$

as under the identification $i = 2u + 1, j = v$, we have

$$\begin{aligned} &\mathrm{Symd}(B_K, \tau) \\ &\quad = \langle 1\otimes 1, u\otimes 1, v\otimes 1, 2uv\otimes u + v\otimes u + uv\otimes 1, 2uv\otimes v + v\otimes v, \\ &\qquad\quad 4uv\otimes uv + 2v\otimes uv + 2uv\otimes v + v\otimes v\rangle \\ &\quad = \langle 1\otimes 1, u\otimes 1, v\otimes 1, 2uv\otimes u + v\otimes u + uv\otimes 1, 2uv\otimes v + v\otimes v, \\ &\qquad\quad 2uv\otimes uv + v\otimes uv\rangle, \\ &\mathrm{Symd}(B_K, \tau') \\ &\quad = \langle 1\otimes 1, u\otimes 1, 2uv\otimes 1 + v\otimes 1, 2v\otimes u + v\otimes 1, v\otimes v, v\otimes uv\rangle \\ &\quad = \langle 1\otimes 1, u\otimes 1, 2uv\otimes 1 + v\otimes 1, v\otimes u - uv\otimes 1, v\otimes v, v\otimes uv\rangle. \end{aligned}$$

This follows (if $\mathrm{ind}_K(B_K) = 4$ and so B_K is a division algebra) also by a theorem of Renard-Tignol-Wadsworth [RTW, Prop 3.13 (ii), Prop 3.15]. (Use (ibid., Rem 2.4) to see that v is defectless.)

3.1.4 Lifting the invariant

We now continue the work of §3.1.2 in the wild case. Throughout this section, let (K, R, k) be a 2-triple and A a biquaternion algebra over k with lifted Azumaya algebra B over R. Now $\tilde{\rho}_{\mathrm{S06}}$ and ρ_{BI} have different value groups, so we first give some remarks on how they relate and how we can use the uniqueness statement of Theorem 2.37 to compare the invariants.

(a) Preparing the ingredients – By a theorem of Kato, we have an isomorphism $\psi_k : H_2^4(k) \to \overline{I^3W_q(k)}$ [Kat2]. Similar to Suslin's construction (1.13), we can also give a morphism $H_{4,A}^4(k) \to H_2^4(k)$. Indeed, the *projection*

$$\pi_1^2 : W_2(k) \to W_1(k) : (a_0, a_1) \to (a_0)$$

gives a morphism $r : H_4^4(k) \to H_2^4(k)$. Since π_1^2 sends elements of order 2 to 0, r does exactly the same. Hence we get a morphism $r_A : H_{4,A}^4(k) \to H_2^4(k)$ because any element of $K_2(k) \cdot [A]$ is of order 2. Now we can compare the different groups with a commutative diagram.

Proposition 3.8

For any 2-extension (K', R', k') of (K, R, k), the following diagram commutes:

$$\begin{tikzcd}
H_{4,A}^4(k') \arrow[r, "r_A"] \arrow[d, "i^*"] & H_2^4(k') \arrow[r, "\psi_{k'}", "\cong"'] \arrow[d, "i^*"] & \overline{I^3W_q(k')} \arrow[d, "j"] \\
H_{4,B_K}^4(K') \arrow[r, "r_B"'] & H_2^4(K') \arrow[r, "\cong", "\psi_{K'}"'] & \overline{I^3W_q(K')}.
\end{tikzcd} \tag{3.2}$$

Remark 3.9 – The morphisms $r_B = r_{B_{K'}}$ and $\psi_{K'}$ are as in (1.13) and (3.1), while $r_A = r_{A_{k'}}$ and $\psi_{k'}$ are as above. The morphism j on Witt groups is as in [Bae, Ch. V, Cor. 1.5]; it is the composition of a bijection of $W_q(R') \cong W_q(k')$ induced by the residual morphism $R' \to k'$ and an injection $W_q(R') \to W_q(K')$. Here, $W_q(R')$ is the Witt group of quadratic spaces of *constant rank* over R'. See [Bae, Ch. I and V] for more information. The maps i^* are defined by Kato as in Remark 2.22 and Proposition 2.31.

Proof. Let $(K'_{\mathrm{nr}}, R'_{\mathrm{nr}}, k'_s)$ be a 2-triple associated with k'_s. So R'_{nr} is the integral closure of R' in K'_{nr}.

We first prove $i^* \circ r_A = r_B \circ i^*$. This follows merely by the definition of i^*. Let $(a_0, a_1) \otimes x_1 \otimes x_2 \otimes x_3 \in H^4_{4,A}(k')$ and take $(b_0, b_1) \in W_2(k'_s)$ such that $(b_0^2, b_1^2) - (b_0, b_1) = (a_0, a_1)$. Then $(a_0) = (b_0)^2 - (b_0) \in W_1(k')$ and

$$i^* \circ r_A((a_0, a_1) \otimes x_1 \otimes x_2 \otimes x_3) = (\bar\sigma(b_0) - b_0)_{\sigma \in \Gamma_{K'}} \cup h^3_2(\{x_1, x_2, x_3\}),$$

where we consider $\bar\sigma(b_0) - b_0$ as an element of $\mathbb{Z}/2\mathbb{Z}$ for any $\sigma \in \Gamma_{K'}$ (with residue $\bar\sigma \in \Gamma_{k'}$). On the other hand,

$$\begin{aligned} r_B \circ i^*((a_0, a_1) \otimes x_1 \otimes x_2 \otimes x_3) & \\ &= r_B\left[(\bar\sigma(b_0, b_1) - (b_0, b_1))_{\sigma \in \Gamma_{K'}} \cup h^3_4(\{x_1, x_2, x_3\})\right] \\ &= (\bar\sigma(b_0) - (b_0))_{\sigma \in \Gamma_{K'}} \cup h^3_2(\{x_1, x_2, x_3\}). \end{aligned}$$

The commutativity of the right square is essentially due to Kato [Kat2, Lem. 11]. He proves the existence of a commutative diagram

$$\begin{array}{ccc} H^n_2(k') & \xrightarrow{\cong} & \overline{I^3 W_q(k')} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle j} \\ K_n(K')/2K_n(K') & \xrightarrow[\psi^1_{K'}]{\cong} & \overline{I^3 W_q(K')}, \end{array}$$

where $\psi^1_{K'}$ is the isomorphism of Milnor's conjecture on quadratic forms (see §3.1.2) and where φ is defined by

$$\bar b \frac{d\bar a_1}{\bar a_1} \wedge \frac{d\bar a_2}{\bar a_2} \wedge \frac{d\bar a_3}{\bar a_3} \mod I \mapsto \{1 + 4b, a_1, a_2, a_3\} \mod 2K_n(K'),$$

for $a_1, a_2, a_3, b \in R'$. Since the isomorphism $\psi_{K'} : H_2^4(K') \to \overline{I^3W_q(K')}$ is defined as composition of $\psi^1_{K'}$ with the Galois symbol $h^4_{2,K'}$, it suffices to check $i(\bar{b}) = h^1_{2,k'}(\{1+4b\})$ for any $b \in R'$. So take $c \in k'_s$ such that $c^2 - c = \bar{b}$. Then

$$i(\bar{b}) = (\bar{\sigma}(c) - c)_{\sigma \in \Gamma_{K'}} \in H^1(K', \mathbb{Z}/2).$$

Take $\tilde{c}$ to be a lift of c in R_{nr}. After change of the representant of $\bar{b}$ in R', we can assume $\tilde{c}^2 - \tilde{c} = b$. Then $1 + 4b = (2\tilde{c} + 1)^2$ and

$$h^1_{2,K'}(\{1 + 4b\}) = (\sigma(2\tilde{c} + 1)/(2\tilde{c} + 1))_{\sigma \in \Gamma_{K'}} \in H^1_2(K').$$

So if $\sigma(2\tilde{c} + 1)/(2\tilde{c} + 1) = 1$, we have $\sigma(\tilde{c}) = \tilde{c}$. On the other hand, if $\sigma(2\tilde{c}+1)/(2\tilde{c}+1) = -1$, we get $\sigma(\tilde{c}) = -\tilde{c} - 1$. This gives indeed the desired equality. ■

(b) Cooking up the result – Using Theorem 2.37 and Proposition 3.8, we can prove the main theorem.

Theorem 3.10

Let k be a field of characteristic 2 and A a biquaternion algebra over k, then

$$\rho_{\mathrm{BI},A} = \psi \circ r_A \circ \tilde{\rho}_{\mathrm{S06},A}$$

with ψ and r_A as in (3.2).

Proof. Let (K, R, k) be a 2-triple associated with k and let (K', R', k') be any 2-extension of (K, R, k). Suppose σ is a symplectic involution on A and take B a lifted Azumaya R-algebra with lifted symplectic involution τ. We use the morphisms from Proposition 3.8. We know j is injective (Remark 3.9), $i^* \circ \tilde{\rho}_{\mathrm{S06},A} = \rho_{\mathrm{S06},B_K}$ (by definition of $\tilde{\rho}_{\mathrm{S06},A}$), and $\rho_{\mathrm{BI},B_K} = \varphi \circ \pi^* \circ \rho_{\mathrm{S06},B_K}$ (§3.1.2). So it suffices to prove that $\rho_{\mathrm{BI},B_K} = j \circ \rho_{\mathrm{BI},A}$.

Suppose $\mathbf{SK}_1(A)(k') \neq 0$. This means $\mathrm{ind}_k(A) = \mathrm{ind}_K(B_K) = 4$, since otherwise $\mathbf{SK}_1(A) = 0 = \mathbf{SK}_1(B_K)$ by Theorem I.20. Also $\mathrm{ind}_{k'}(A_{k'}) = \mathrm{ind}_{K'}(B_{K'}) = 4$, so we get that $A_{k'}$ and $B_{K'}$ are division algebras. Then $B_{K'}$ is equipped with a valuation w (see §2.1.2 (a)). Recall that the associated valuation ring is $B_{R'}$ with reduced k-algebra $A_{k'}$, that $\mathbf{SL}_1(B_K)(K')$ is part of $B_{R'}$, and that the isomorphism $\mathbf{SK}_1(B_K)(K') \cong \mathbf{SK}_1(A)(k')$ is induced by the residue map on $\mathbf{SL}_1(B_K)(K')$.

In this case σ and τ cannot be hyperbolic due to [KMRT, Prop. 6.7 (3)]. Take $a \in \mathbf{SK}_1(A)(k')$ with lift $b \in \mathbf{SK}_1(B_K)(K')$. Then by definition it follows that $\mathrm{Prd}_{A,a/k'}(X) = \overline{\mathrm{Prd}_{B,b/K'}(X)}$, where the residue is the canonical residue on $R'[X]$. So we also get $\mathrm{Prp}_{\sigma,a/k'}(X) = \overline{\mathrm{Prp}_{\tau,b/K'}(X)}$ and $\mathrm{Trp}_{\sigma/k'}(a) = \overline{\mathrm{Trp}_{\tau/K'}(b)}$. Now take $y \in \mathrm{Symd}(B_{K'}, \tau_{K'}) \cap B_{K'}^\times$ satisfying $y(\mathrm{Trp}_{\tau/K'}(y) - y)^{-1} = -\tau(b)b$. We can assume $w(y) \geq 0$, since if $w(y) < 0$, i.e. $\mathrm{Nrd}_{B_{K'}/K'}(y) = \lambda/\mu \in K'$ with $\lambda, \mu \in R'$, then $w(\mu y) = v(\lambda) \geq 0$ and

$$\mu y \left(\mathrm{Trp}_{\tau/K'}(\mu y) - \mu y\right)^{-1} = y(\mathrm{Trp}_{\tau/K'}(y) - y)^{-1}.$$

Hence for $w(y) \geq 0$, we get $\bar{y}(\mathrm{Trp}_{\sigma/k'}(\bar{y}) - \bar{y})^{-1} = -\sigma(a)a$ because b is a lift of a. Moreover, clearly $\bar{y} \in \mathrm{Symd}(A, \sigma)$.

Then

$$\begin{aligned} \rho_{\mathrm{BI},A,k'}(a) &= \Phi_{\bar{y}} : A_{k'} \to k' : x \mapsto \mathrm{Trp}_{\sigma/k'}(\sigma_{k'}(x)\bar{y}x) \quad \text{and} \\ \rho_{\mathrm{BI},B_{K'},K'}(b) &= \Phi_y : B_{K'} \to K' : x \mapsto \mathrm{Trp}_{\tau/K'}(\tau_{K'}(x)yx). \end{aligned}$$

Since for $x \in B$, we have $\overline{\mathrm{Trp}_{\tau/K'}(\tau_{K'}(x)yx)} = \mathrm{Trp}_{\sigma/k'}(\sigma_{k'}(\bar{x})\bar{y}\bar{x})$, we get the required compatibility. ■

(c) Non-triviality of the invariant – Because the invariants for biquaternions in characteristic zero are injective, they are also injective in characteristic 2 due to the lifting property (Theorem 2.37). As $\mathbf{SK}_1$ is not trivial for Platonov's examples (Example I.10) and in general for biquaternion algebras of index 4 (Theorem I.20), we retrieve non-trivial invariants in characteristic 2.

Another argument for non-triviality of ρ_{BI} in characteristic different from 2 is given by a formula of Merkurjev for the value on the centre of the biquaternion algebra [Mer2, Ex. p. 70] – see also [KMRT, Ex. 17.23]. Using this formula and the lift from characteristic 2 to characteristic 0, one could hope to prove the non-triviality of ρ_{BI} (and hence of $\tilde{\rho}_{\mathrm{S06}}$) in the case when $\mathrm{char}(k) = 2$, but this fails. Let us comment on this fact.

Let (K, R, k) be a 2-triple and let $A = [\bar{a}, \bar{b}) \otimes_k [\bar{c}, \bar{d})$ be a biquaternion k-algebra for $a, c \in R$ and $b, d \in R^\times$. Then the lifted Azumaya R-algebra is $B = (4a+1, b)_R \otimes_R (4c+1, d)_R$ (see §3.1.3 (b)). Suppose K contains a primitive fourth root of unity ζ, then by (loc. cit.) we have

$$\rho_{\mathrm{BI},B_K,K}([\zeta]) = \langle\langle 4a+1, b, 4c+1, d\rangle\rangle + I^4 W_q'(K),$$

where $[\zeta]$ is the class of ζ in $\mathbf{SK}_1(B_K)(K)$.

Let π be the isomorphism $\mathbf{SK}_1(B_K)(K) \cong \mathbf{SK}_1(A)(k)$, then $\pi([\zeta]) = [1]$ because k contains no non-trivial fourth roots of unity. By the proof of Theorem 3.10, we have $j \circ \rho_{\mathrm{BI},B_K,K}([\zeta]) = \rho_{\mathrm{BI},A,k} \circ \pi([\zeta]) = 0 \in \overline{I^3W'_q(k)}$. Because the map j from Proposition 3.8 is injective, we get that $\langle\langle 4a+1, b, 4c+1, d\rangle\rangle = 0 \in \overline{I^3W'_q(K)}$. We can also verify this by calculating with Pfister forms. Define $\mathcal{Q}$ as the symbol R-algebra $(4a+1, b)$ and let $\mathcal{X}$ be the natural affine R-scheme with

$$\mathcal{X}(R) = \{x \in \mathcal{Q} \,|\, \mathrm{Nrd}_{\mathcal{Q}_K/K}(x) = 4c+1\},$$

where $\mathcal{Q}_K = \mathcal{Q} \otimes_R K$. Then $\mathcal{X}$ is an R-torsor under $\mathbf{SL}_1(\mathcal{Q})$, where $\mathbf{SL}_1(\mathcal{Q})$ is the natural affine R-scheme so that $\mathbf{SL}_1(\mathcal{Q})(R) = \mathbf{SL}_1(\mathcal{Q}_K)(K) \cap \mathcal{Q}$. The special fibre $\mathcal{X}_k = \mathcal{X} \times_R k$ clearly has a rational point, so its class $[\mathcal{X}_k] \in H^1(k, \mathbf{SL}_1(\mathcal{Q}_k))$ is trivial. By Hensel's lemma à la Grothendieck, we get $[\mathcal{X}] = 0 \in H^1_{\text{ét}}(R, \mathbf{SL}_1(\mathcal{Q}))$. Hence $\mathcal{X}$ (as well as the generic fibre $\mathcal{X}_K$) has a rational point, but then by theory of Pfister forms we get $\langle\langle 4a+1, b, 4c+1\rangle\rangle = 0 \in W'_q(K)$ [Kah2, Cor. 2.1.10]. Indeed, $\mathrm{Nrd}_{\mathcal{Q}_K/K}(x)$ corresponds with a value of $\langle\langle 4a+1, b\rangle\rangle$. So a fortiori $\langle\langle 4a+1, b, 4c+1, d\rangle\rangle = 0 \in \overline{I^3W'_q(k)}$.

3.2 Kahn's invariant

We compare now all defined invariants of $\mathbf{SK}_1(A)$ to $\rho_{\mathrm{Kahn},A}$ in the moderate case, i.e. as they are originally defined. The results can be generalised to the wild invariants, but with some loss of information. We also generalise the formula of Merkurjev (§3.1.4 (c)) for the value on the centre of biquaternion algebras to the tensor product of two symbol algebras.

For sake of convenience, we also use the following terminology.

Definition 3.11

Suppose ρ is an invariant of $\mathbf{SK}_1$ which is defined for any central simple algebra A with index n not divisible by the characteristic of its base field and which has values in the Galois cohomology group $\mathcal{H}^4_{n,\mathcal{A}^{\otimes r}}$ for r a fixed integer. Then we say ρ is a *moderate invariant of $\boldsymbol{SK}_1$ with values in $\mathcal{H}^4_{\otimes r}$*. We denote by ρ_A the invariant for a central simple algebra A.

3.2.1 Moderate case

Let A be a central simple k-algebra with $\mathrm{ind}_k(A) = n \in k^\times$ and $m = \mathrm{per}_k(A)$. We explain two natural ways of comparing the invariant groups $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*)$ and $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$.

(a) Ways of looking – For any field extension F of k and any integer r, we can look at the composition

$$m_r : H^4_{n,A^{\otimes r}}(F) \xrightarrow{\cdot m} H^4_{n/m}(F) \to H^4_n(F)$$

and at the projection

$$\pi_r : H^4_n(F) \to H^4_{n,A^{\otimes r}}(F).$$

These induce respectively maps

$$\begin{aligned} \tilde{m}_r : \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}) &\to \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*) \quad \text{and} \\ \tilde{\pi}_r : \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*) &\to \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}). \end{aligned}$$

The maps $\tilde{\pi}_r$ where introduced by Kahn [Kah3, Rem. 11.6], but we rather consider the maps $\tilde{m}_r$ to compare because of the special definition of Kahn's invariant as generator of the the target group. We could also refine $\tilde{m}_r$ if $H^2(k, \mu_n^{\otimes 2}) \cup r[A]$ has m'-torsion for an integer $0 \leq m' < m$. A good comprehension of both maps actually relies, as Kahn mentions, on a good comprehension of the cup product with the class of A (loc. cit.).

By the cyclicity of $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*)$ (§1.4 (d)), we certainly find the following relations. Recall the definition of the integer $\overline{n}$ retrieved from an integer n (§1.4 (d)).

Proposition 3.12

Let A be a central simple k-algebra with $\mathrm{ind}_k(A) = n \in k^\times$. Then for any integer r and any $\rho \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}})$, there exists an integer $d_A \in \mathbb{Z}/\overline{n}$ such that

$$\tilde{m}_r(\rho) = d_A\, \rho_{\mathrm{Kahn},A} \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*) \subset \mathbb{Z}/\overline{n}.$$

Proof. Use the definition of ρ_{Kahn} and the bounds on $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}_n^*)$ (see §1.4 (d)). ∎

Kahn also raises the issue whether $\tilde{\pi}_r$ is surjective or not (loc. cit.). We can prove it to be non-surjective for biquaternion division algebras à la Platonov.

Proposition 3.13

Let $k = \mathbb{Q}_p((t_1))((t_2))$ for a prime p. Suppose $A = (a, t_1) \otimes (b, t_2)$ is a biquaternion division k-algebra for $a, b \in \mathbb{Q}_p^\times$. Then $\tilde{\pi}_1$ is not surjective.

Proof. In Example I.10 we saw that $\mathrm{SK}_1(A) \cong \mathbb{Z}/2$. Using (1.4), $\mathrm{cd}(\mathbb{Q}_p) = 2$, and $\mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ [Ser2, Ch. II, §5.1 & Prop. 15], we find that $H_4^4(k) \cong \mathbb{Z}/4$. We can also add a fourth primitive root of unity to k as this does not change the Brauer group. In this case we have the Bloch-Kato isomorphism $H_4^4(k) \cong K_4(k)/4$.

We now prove $H_{4,A}^4(k) \cong \mathbb{Z}/2$. Under the Bloch-Kato isomorphism $K_2(k)/2 \cong {}_2\mathrm{Br}(k)$, the class of A corresponds to $\{a, t_1\} + \{b, t_2\} \in K_2(k)/2$ (§1.1 (b)) so that $H^2(k, \mu_4^{\otimes 2}) \cup [A]$ is isomorphic to $(K_2(k)/4) \cdot (2\{a, t_1\} + 2\{b, t_2\})$. As the isomorphism $H_4^4(k) \cong \mathbb{Z}/4$ is retrieved by taking two residues $\partial_{t_1}^3$ and $\partial_{t_2}^4$, it suffices to determine the group (cfr. (1.10))

$$\partial_{t_1}^3 \circ \partial_{t_2}^4 \big((K_2(k)/4) \cdot (2\{a, t_1\} + 2\{b, t_2\})\big).$$

By the definition of residues on Milnor K-groups [Mil5, §2], it is clear that this equals $(K_1(\mathbb{Q}_p)/4) \cdot 2\{a\} + (K_1(\mathbb{Q}_p)/4) \cdot 2\{b\}$. As we assumed that $\mathbf{SK}_1(A)$ is not trivial, a cannot be a square by Wang's theorem. This means that $(K_1(\mathbb{Q}_p)/4) \cdot 2\{a\} + (K_1(\mathbb{Q}_p)/4) \cdot 2\{b\}$ is not trivial. On the other hand it has 2-torsion inside $K_2(\mathbb{Q}_p)/4 \cong \mathbb{Z}/4$ so that indeed $H_{4,A}^4(k) \cong \mathbb{Z}/2$.

Then $\pi_1 : \mathbb{Z}/4 \to \mathbb{Z}/2$ is the "modulo 2" map and $m_1 : \mathbb{Z}/2 \to \mathbb{Z}/4$ is the canonical injection. Suslin proves $\rho_{\mathrm{S06},A,k} : \mathbf{SK}_1(A)(k) \to H_{4,A}^4(k)$ is not trivial (1.13), so it is the identity map on $\mathbb{Z}/2$. It is then clear that this can never factor through $H_4^4(k)$ so that $\tilde{\pi}_1$ is clearly not surjective. ∎

(b) Determining factors – We prove that for the product of two symbol algebras of degree n the factor, d_A appearing in Proposition 3.12 only depends on the invariant ρ and the characteristic of k.

Proposition 3.14

Let ρ be a moderate invariant of $\mathbf{SK}_1$ with values in $\mathcal{H}^4_{\otimes r}$. Let furthermore p be equal to zero or to any prime and let m be an integer not divisible by p. Then there exist an integer $i(p,m) \in \mathbb{Z}/\overline{m^2}$ such that for any field k of $\mathrm{char}(k) = p$ containing a primitive m-th root of unity ξ_m and for any product $A = (a,b)_m \otimes (c,d)_m$ of two symbol k-algebras

$$\tilde{m}_r(\rho_A) = i(p,m)\, \rho_{\mathrm{Kahn},A} \in \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{m^2}) \subset \mathbb{Z}/\overline{m^2}.$$

Remark 3.15 – Although $i(p,m)$ is in general not uniquely determined, we can take a canonical representant as we know $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{m^2})$ is cyclic. This comes down to taking the class in $\mathbb{Z}/\overline{m^2}$ satisfying the required relation and such that the representant in $\{0, \ldots, \overline{m^2} - 1\}$ is as low as possible. It also of course depends on the invariant. We add an index if necessary to stress which invariant is compared to Kahn's invariant. Moreover, it also depends on the exact definition of the injection $\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{m^2}) \subset \mathbb{Z}/\overline{m^2}$, but this can be chosen in a canonical way due to the results of Kahn [Kah3, Def. 11.3].

Proof. Take k the prime field of characteristic p and set $k' = k(\xi_m)$ for an m-primitive root of unity $\xi_m \in k_s$. Denote by $\mathcal{T} = (t_1,t_2)_m \otimes (t_3,t_4)_m$ the product of two Azumaya symbol algebras over $R = k'[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]$ where t_1, t_2, t_3, t_4 are variables and where Azumaya symbol algebras are defined using the same relations as used for symbol algebras over a field. Take $K = k'(t_1,t_2,t_3,t_4)$ and $T = \mathcal{T}_K = (t_1,t_2)_m \otimes (t_3,t_4)_m$, the product of the respective symbol algebras over K. By Proposition 3.12, we find a unique $d_T \in \mathbb{Z}/\overline{m^2}$ such that

$$\tilde{m}_r(\rho_T) = d_T\, \rho_{\mathrm{Kahn},T}. \tag{3.3}$$

We prove d_T only depends on m and p.

So suppose F is a field of characteristic p containing an m-th primitive root of unity so that $k' \subset F$. Take any product $A = (a,b)_m \otimes (c,d)_m$ of two symbol algebras of degree m over F. Now A can be obtained from $\mathcal{T}_F = \mathcal{T} \otimes_R F$ by specialising t_1, t_2, t_3, t_4 to a, b, c, d respectively.

Moreover, (a,b,c,d) defines a k-rational point x of $\mathrm{Spec}(F[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}])$. Take $\mathcal{O}_x$ to be the local ring of $\mathrm{Spec}(F[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}])$ in x with maximal ideal M. It is clear that the completion $\hat{\mathcal{O}}_x$ of $\mathcal{O}_x$ with respect to the M-adic topology is F-isomorphic to $R' = F[[u_1, u_2, u_3, u_4]]$ where $u_1 = t_1 - a, u_2 = t_2 - b, u_3 = t_3 - c$, and $u_4 = t_4 - d$ (see also [Gro1, Thm. 19.6.4]). Under the isomorphism $\mathrm{Br}(R') \cong \mathrm{Br}(F)$ from §2.1.2 (a), it is clear that $A_{R'} = A \otimes R'$ is an Azumaya R'-algebra mapping to A. Furthermore, the F-isomorphism of $\hat{\mathcal{O}}_x$ with R' gives an isomorphism $\mathrm{Br}(\hat{\mathcal{O}}_x) \cong \mathrm{Br}(R')$. In its turn, this gives an isomorphism $\mathrm{Br}(\hat{\mathcal{O}}_x) \to \mathrm{Br}(F)$, with inverse given by taking the tensor product over F with $\hat{\mathcal{O}}_x$. It sends the class of $\mathcal{T}_{\hat{\mathcal{O}}_x}$ to the class of A.

Let $K' = F((u_1))((u_2))((u_3))((u_4))$, then $A \otimes_F K'$ is Brauer-equivalent to $\mathcal{T}_{\hat{\mathcal{O}}_x} \otimes_{\hat{\mathcal{O}}_x} K' \cong T_{K'}$. By Corollary 2.3, $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(T_{K'})$. Furthermore, (1.4) gives an injection $H^4_{m^2}(F) \to H^4_{m^2}(K')$. The diagram

$$\begin{tikzcd}
\mathrm{SK}_1(A) \arrow[r, "\rho"] \arrow[d, "\cong"'] & H^4_{m^2}(F) \arrow[d] \\
\mathrm{SK}_1(T_{K'}) \arrow[r, "\rho"'] & H^4_{m^2}(K')
\end{tikzcd}$$

commutes for both $\tilde{m}_r(\rho)$ and ρ_{Kahn} (by definition of an invariant). Then by (3.3) and functoriality of the arguments, we get $\tilde{m}_r(\rho_A) = d_T \rho_{\mathrm{Kahn},A}$. ■

(c) Non-triviality of Kahn's invariants – As mentioned in Remark 1.11, ρ_{Kahn} is not-trivial for biquaternion algebras (of index 4). We generalise this to the product of two cyclic algebras à la Platonov (Ex I.10). For that purpose, we compare ρ_{Kahn} to ρ_{S91} as this invariant is non-trivial for Platonov's examples (§1.4 (a)). This means that we have to work with $\mathcal{H}^*_{n,\mathcal{A}^{\otimes 2}}$ for suitable n and A. (In the same way as in Proposition 3.13, these give also examples of non-trivial $\tilde{\pi}_2$.)

Theorem 3.16

Let k be p-adic field containing a n^3-th primitive root unity and let $F = k((t_1))((t_2))$. Suppose $A = (a, t_1)_n \otimes (b, t_2)_n$ is a division F-algebra, then $\rho_{\mathrm{Kahn},A}$ is not trivial. If $n = q_1 \cdot \ldots \cdot q_r$ for different primes q_i, then

$$\mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n^2}) \cong \mathbb{Z}/n.$$

Moreover if n is odd, the integer $i_{\mathrm{S91}}(0, n) \in \mathbb{Z}/n^2$ defined in Proposition 3.14 for ρ_{S91} is not trivial.

Proof. We know $\mathrm{SK}_1(A) \cong \mathbb{Z}/n$ by Example I.10. Furthermore $H^4_{n^2}(F) = \mathbb{Z}/n^2$ (arguments as in the proof of Proposition 3.13).

To calculate $H^4_{n^2,A^{\otimes 2}}(F)$, we use an analogous argument as in the proof of Proposition 3.13. If n is odd, we also find $H^4_{n^2,A^{\otimes 2}}(F) \cong \mathbb{Z}/n$ as in this case $\mathrm{per}_k(A^{\otimes 2}) = \mathrm{per}_k(A)$. If n is even, $\mathrm{per}_k(A^{\otimes 2}) = n/2$ so that $H^4_{n^2,A^{\otimes 2}}(F) \cong \mathbb{Z}/(2n)$. In either case, $m_2 : H^4_{n^2,A^{\otimes 2}}(F) \to H^4_{n^2}(F)$ is the canonical injection (m_2 is the multiplication by m for $m = n$ if n odd and $m = n/2$ if n even).

Suslin proves $\rho_{\mathrm{S91},A}$ is not trivial (on the field F) [Pla, Thm. 4.8]. If n is odd, $\rho_{\mathrm{Kahn},A}$ is not trivial (on F) by Proposition 3.12 and hence by definition $i_{\mathrm{S91}}(0, n^2) \neq 0 \in \mathbb{Z}/\overline{n^2}$. If n is even, a similar argument as in the proof of Proposition 3.12 gives the non-triviality of $\rho_{\mathrm{Kahn},A}$ (mutatis mutandis m by $n/2$).

By the bound on the invariant group (§1.4 (d)) and a Brauer decomposition of A with a related decomposition of invariants in primary parts, the isomorphism statement follows. ∎

3.2.2 Wild case

Now, we continue the comparison in the wild case. Using a lift, we can generalise the statement to any central simple algebra with some loss of information. This does let us prove a relation between the several $i(p, n)$'s.

Let A be a central simple k-algebra of $\mathrm{ind}_k(A) = n$ and $\mathrm{per}_k(A) = m$. We define the functors of graded groups for r an integer

$$\mathcal{H}^*_n : k\text{-}\mathfrak{fields} \to \mathfrak{Groups} \quad : \quad F \mapsto (H^i_n(F))_{i>0}, \text{ and}$$

$$\mathcal{H}^*_{n,\mathcal{A}^{\otimes r}} : k\text{-}\mathfrak{fields} \to \mathfrak{Groups} \quad : \quad F \mapsto (H^i_n(F)/(K_{i-2}(F) \cdot r[A_F])_{i\geq 2}.$$

They are in general no cycle module as to obtain a cycle module we have to add in an extra field L (see Definitions 2.23 & 2.28).

We again have a morphism

$$\tilde{m}_r : \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_{n,\mathcal{A}^{\otimes r}}) \to \mathrm{Inv}^4(\mathbf{SK}_1(A), \mathcal{H}^*_n),$$

induced by the multiplication for any field extension F of k:

$$m_r : H^4_{n,A^{\otimes r}}(F) \xrightarrow{\cdot m} H^4_{n/m}(F) \to H^4_n(F).$$

Note that we can also define a map $\tilde{\pi}_r$ as in §3.2.1 (a).

Proposition 3.17

> Let ρ be a moderate invariant of $\mathbf{SK}_1$ with values in $\mathcal{H}^4_{\otimes r}$. Suppose k is a field of $\mathrm{char}(k) = p > 0$ and let $A = [a,b)_p \otimes [c,d)_p$ be the product of two p-algebras over k, then
>
> $$\tilde{m}_r(\tilde{\rho}_A) = i(0,p)\,\tilde{\rho}_{\mathrm{Kahn},A}.$$

Proof. Let (K,R,k) be a p-ring. The lifted Azumaya R-algebra B of A is (after base extension to K) a product of two symbol algebras of degree p. This follows from the injection $H^2_{p^2}(k) \to H^2_{p^2}(K)$ (see Remark 2.22) and from the description of the image of A and B_K in the second cohomology groups as described in Remarks 1.2 and 2.20.

The result follows immediately from the injections

$$\begin{aligned} \mathrm{Inv}^4(\mathbf{SK}_1(B_K),\mathcal{H}^*_{p^2}) &\to \mathrm{Inv}^4(\mathbf{SK}_1(A),\mathcal{H}^*_{p^2}) \quad \text{and} \\ \mathrm{Inv}^4(\mathbf{SK}_1(B_K),\mathcal{H}^*_{p^2,\mathcal{B}_K^{\otimes r}}) &\to \mathrm{Inv}^4(\mathbf{SK}_1(A),\mathcal{H}^*_{p^2,\mathcal{A}^{\otimes r}}) \end{aligned}$$

defined by lifting invariants (Theorem 2.43) and the relations for ρ_{B_K} and ρ_{Kahn,B_K} (Proposition 3.14). ■

Remark 3.18 – In the view of Remark 2.19, we could even refine the statement in the moderate case. Let (K,R,k) be a p-triple and $A = (a,b)_n \otimes (c,d)_n$ a product of two symbol k-algebras for $n \in k^\times$, then a similar statement holds as A lifts to the central simple K-algebra $(\tilde{a},\tilde{b})_n \otimes (\tilde{c},\tilde{d})_n$ where $\tilde{a},\tilde{b},\tilde{c},\tilde{d} \in R$ are lifts from a,b,c,d (see Remark 1.2 and §1.1 (c)).

If $\tilde{\rho}_A = \rho_A$, then $i(p,n)$ is a multiple of $i(0,n)$ in $\mathbb{Z}/\overline{n}$. Indeed, $\rho_{\mathrm{Kahn},A}$ is a generator of $\mathrm{Inv}^4(\mathbf{SK}_1(A),\mathcal{H}^*_n) \subset \mathbb{Z}/\overline{n}$ and for some integer λ

$$i(p,n)\rho_{\mathrm{Kahn},A} = \tilde{m}_r(\rho_A) = i(0,n)\,\tilde{\rho}_{\mathrm{Kahn},A} = i(0,n)\,\lambda\,\rho_{\mathrm{Kahn},A}.$$

In particular, $i(p,n) = i(0,n)$ if $\tilde{\rho}_{\mathrm{Kahn},A} = \rho_{\mathrm{Kahn},A}$ so that the integers $i(p,n)$ would not depend on the characteristic of the base field.

3.2.3 Formula on the centre

We can now generalise the formula of Merkurjev on the centre of a biquaternion algebra ([Mer2, Ex. p.70] – see also [KMRT, Ex. 17.23] and §3.1.4 (c)) to the tensor product of two symbol algebras. We first prove a general formula and later we prove a finer result using Theorem 3.16.

(a) General result – We again use cohomological invariants, however not invariants of algebraic groups as in §1.3, but rather invariants as introduced in [GMS, Ch. I]. These are also natural transformations of functors, but rather a natural transformation of a functor $B : k\text{-}\mathfrak{fields} \to \mathfrak{Sets}$ into a functor $H : k\text{-}\mathfrak{fields} \to \mathfrak{Ab}$. For the natural transformation cause, we consider H to be a functor $k\text{-}\mathfrak{fields} \to \mathfrak{Sets}$.

Proposition 3.19

Let p be equal to 0 or to any prime and let $n > 0$ be an integer not divisible by p. There exists an integer $j(p, n)$ such that the following formula holds for any field k of $\mathrm{char}(k) = p$ containing a primitive n^2-th root of unity ζ and for $A = (a, b)_n \otimes (c, d)_n$ any product of two symbol k-algebras (for $a, b, c, d \in k^\times$):

$$\rho_{\mathrm{Kahn},A,k}([\zeta]) = \varphi\left[j(p, n)\, h^4_{m,k}(\{a, b, c, d\})\right] \in H^4_{n^2}(k).$$

Here, φ is the canonical map $H^4_m(k) \to H^4_{n^2}(k)$ (for $m = \overline{n^2}$).

Remark 3.20 – Remark that $\mu_{n^2}^{\otimes i} \cong \mathbb{Z}/n^2$ as Γ_k-modules for any $i > 0$ as k contains an n^2-th primitive root of unity. Note also that $\varphi[h^4_{m,k}(\{a, b, c, d\})] = m'\, h^4_{n^2,k}(\{a, b, c, d\})$ for $m' = n^2/m$ and that that φ is injective. The former follows from the definitions and the latter follows from the long exact sequence in Galois cohomology associated with

$$0 \to \mathbb{Z}/m \to \mathbb{Z}/n^2 \to \mathbb{Z}/m' \to 0,$$

which by the Bloch-Kato isomorphism comes down to

$$\ldots \to K_3(k)/n^2 \to K_3(k)/m' \to K_4(k)/m \xrightarrow{\varphi} K_4(k)/n^2 \ldots$$

Now, $K_3(k)/n^2 \to K_3(k)/m'$ is clearly surjective so that φ is indeed injective.

Remark 3.21 – This expression is indeed compatible with the biquaternion case keeping in mind diagrams (1.13) and (3.1). Also, the integer $j(p,n)$ in the theorem is not uniquely determined, but can be picked canonically by taking the smallest positive integer satisfying the relation. Moreover $j(p,n)$ depends on the n-th primitive root of unity used in the definition of the symbol algebra and of the choice of n^2-th primitive root of unity ζ. We are interested in the invertibility of $j(p,n)$ modulo m and therefore the exact choices do not matter, so we do not incorporate them in the notation.

Proof. As ρ_{Kahn} has m-torsion (Lemma 1.10), we can assume $\rho_{\mathrm{Kahn},A,k}([\zeta])$ to have values in $H^4_m(k)$.

Let k be the prime field of characteristic p and set $k' = k(\zeta)$ for $\zeta \in \bar{k}$ a primitive n^2-th root of unity. Take $T = (t_1,t_2)_n \otimes (t_3,t_4)_n$ over $F = k'(t_1,t_2,t_3,t_4)$. We prove the formula for T. The proof ends by specialising to A as in the proof of Proposition 3.14.

Let $B : k\text{-}\mathfrak{fields} \to \mathfrak{Sets}$ be the functor attaching to a field extension F of k the Galois cohomology group $H^1(F,\mu_m)^4$ and H associating $H^4(F,\mu_m^{\otimes 4})$ with F. Then ρ_{Kahn} induces a cohomological invariant of B into H. Indeed, using the isomorphism $H^1(F,\mu_m) \cong F^\times/(F^\times)^m$, we associate with any four representants $a,b,c,d \in F^\times$ of classes in $H^1(F,\mu_m)$ the value $\rho_{\mathrm{Kahn},A,F}([\zeta]) \in H^4_m(F) \cong H^4(F,\mu_m^{\otimes 4}) \cong K_4(F)/m$ (for $A = (a,b)_n \otimes (c,d)_n$).

Using a full description of all possible invariants of B into H of [Gar, Prop. 2.1 & §3.1] and [GMS, Ex. 16.5], we find that $r_n(\rho_{\mathrm{Kahn},T,F}([\zeta]))$ can be written in $K_4(F)/m$ as sum of pure symbols of the form $\lambda\{z_1,z_2,z_3,z_4\}$ where λ is an integer and each z_i is either a t_j or an element of k. We prove that only $\{t_1,t_2,t_3,t_4\}$ occurs. By specialising t_1 to 1, we obtain $T_1 = (1,t_2)_n \otimes (t_3,t_4)_n$ from T. But then $\mathbf{SK}_1(T_1) = 0$ by Wang's theorem so that $\rho_{\mathrm{Kahn},T_1,F}([\zeta]) = 0$. This induces that for all (non-trivial) pure symbols $\{z_1,z_2,z_3,z_4\}$ appearing in $\rho_{\mathrm{Kahn},T,F}([\zeta])$ one of the z_i has to equal t_1 (as the other ones are zero by the specialisation above). Three other specialisations give the result. ■

Remark 3.22 – In the same way as in Remark 3.18, there is a compatibility between the $j(p,n)$'s. Let k be a field of $\mathrm{char}(k) = p > 0$ containing an n^2-th primitive root of unity ζ and take $A = (a,b)_n \otimes (c,d)_n$ a tensor product of two symbol k-algebras of degree $n \in k^\times$. Take (K,R,k) a p-

triple associated with k, then A lifts again to $B_K = (\tilde{a}, \tilde{b})_l \otimes (\tilde{c}, \tilde{d})_l$ where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in R$ are lifts from a, b, c, d.

Under the injection $H^4_m(k) \to H^4_m(K)$ (for $m = \overline{n^2}$) induced by (1.4), $\varphi[h^4_{m,k}(\{a, b, c, d\})]$ is sent to $\varphi[h^4_{m,K}(\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\})]$ (with an abuse of notation for φ from Proposition 3.19). This follows from a splitting for Milnor's K-Theory (1.7).

Now ζ lifts to a primitive n^2-th root of unity $\tilde{\zeta} \in R$. Then by definition of $\tilde{\rho}_{\mathrm{Kahn},A}$ and Proposition 3.19, it follows that

$$\tilde{\rho}_{\mathrm{Kahn},A}([\zeta]) = \varphi\left[j(0,n)\, h^4_{m,k}(\{a,b,c,d\})\right]. \tag{3.4}$$

On the other hand, by the definition of $\rho_{\mathrm{Kahn},A}$ as a generator

$$\tilde{\rho}_{\mathrm{Kahn},A}([\zeta]) = \lambda\, \rho_{\mathrm{Kahn},A}([\zeta]) = \lambda\, \varphi\left[j(p,n)\, h^4_{m,k}(\{a,b,c,d\})\right]$$

for an integer λ. If $\rho_{\mathrm{Kahn},A} = \tilde{\rho}_{\mathrm{Kahn},A}$, we can again take $j(p,n) = j(0,n)$ so that the integers $j(p,n)$ would not depend on the characteristic.

Remark 3.23 – In wild characteristics (i.e. when $p \mid n$), a formula as above does not make sense as there are no non-trivial p^2-th roots of unity. So similar as in §3.1.4 (c), we cannot generalise this formula to wild invariants by means of a lift.

(b) Non-triviality of factor – We prove the non-triviality of the factor appearing in Proposition 3.19. This uses the non-triviality of ρ_{Kahn} for Platonov's examples (Theorem 3.16). First we recall some notions related to tori. See [CTS1] as a reference for more details.

Denote for a finite separable field extension K of k by $R_{K/k}(\mathbb{G}_m)$ the torus obtained by Weil restriction of scalars from K to k (see e.g. Definition B.1). Denote furthermore the kernel of the multiplication map $R_{K/k}(\mathbb{G}_m) \to \mathbb{G}_{m,k}$ by $R^1_{K/k}(\mathbb{G}_m)$ and the cokernel of the injection $\mathbb{G}_{m,k} \to R_{K/k}(\mathbb{G}_m)$ by $R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$. Furthermore for any k-torus T, we denote by $T(k)/R$ the R-equivalence classes of $T(k)$. The dual $\hat{T}$ of a k-torus T is the character group $\mathrm{Hom}(T, \mathbb{G}_m)$. The dual of $R_{K/k}(\mathbb{G}_m)$ is clearly the free abelian group $\mathbb{Z}[\Gamma]$ for $\Gamma = \mathrm{Gal}(K/k)$. The dual of $R^1_{K/k}(\mathbb{G}_m)$ is then J_Γ, the cokernel of the norm:

$$\mathbb{Z} \to \mathbb{Z}[\Gamma] : a \mapsto \sum_{\gamma_i \in \Gamma} a\gamma_i.$$

The dual of $R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$ is the kernel I_Γ of the augmentation map:

$$\mathbb{Z}[\Gamma] \to \mathbb{Z} : \sum_{\gamma_i \in \Gamma} n_i \gamma_i \mapsto \sum_{\gamma_i \in \Gamma} n_i.$$

Recall that a k-torus F is called *flabby* (*flasque*) if $\hat{F}$ is a flabby Γ_k-module, i.e. $\mathrm{Ext}^1(\hat{F}, P) = 0$ for any permutation Γ_k-module P (for equivalent definitions see ibid., Lem. 1). A flasque resolution of a k-torus T is an exact sequence of k-tori

$$0 \to S \to E \to T \to 0$$

with E quasi-trivial (i.e. $\hat{E}$ is a permutation module) and S flabby. This always exists and if T is split by a field K, then E and S can also be chosen to be split by K.

Theorem 3.24

Let k be a p-adic field containing a n^3-th primitive root of unity and let $F = k((t_1))((t_2))$. If $A = (a, t_1)_n \otimes (c, t_2)_n$ is a division F-algebra, then

$$\rho_{\mathrm{Kahn},A,F}([\zeta]) = \varphi\left[\lambda\, h^4_{m,F}(\{a, t_1, c, t_2\})\right] \in H^4_{n^2}(F)$$

for ζ an n^2-th primitive root of unity, $m = \overline{n^2}$, and an integer $\lambda \not\equiv 0 \mod m$ (and φ as in Proposition 3.19). A fortiori, $j(0, n) \not\equiv 0 \mod m$ for any n.

Proof. We know by Theorem 3.16 that $\rho_{\mathrm{Kahn},A} : \mathbf{SK}_1(A)(F) \to H^4_{n^2}(F)$ is not trivial and moreover $\mathbf{SK}_1(A)(F) \cong \mathbb{Z}/n$ and $H^4_{n^2}(F) \cong \mathbb{Z}/n^2$. We prove that the image of $\mu_{n^2}(F) \cong \mathbb{Z}/n^2$ inside $\mathbf{SK}_1(A)(F)$ is all of $\mathbf{SK}_1(A)(F)$. In that case, $\rho_{\mathrm{Kahn},A}([\zeta])$ is not trivial in $H^4_{n^2}(F)$ (and in $H^4_m(F) \cong \mathbb{Z}/m$) so that $j(0, n) \not\equiv 0 \mod m$.

To prove the statement, let $K = k(\sqrt[n]{a}, \sqrt[n]{b})$ and $\Gamma = \mathrm{Gal}(K/k) \cong \mathbb{Z}/n \times \mathbb{Z}/n$. Then by taking residues on F with respect to t_1 and t_2, Platonov proves $\mathbf{SK}_1(A)(F) \cong \hat{H}^{-1}(\Gamma, K^\times)$ where the cohomology group is a Tate cohomology group (see e.g. [Wei1, Def. 6.2.4]) - also use [Pla, Thms. 4.17 & 5.7] and [Wad, (6.15)]). On the other hand, $\hat{H}^{-1}(\Gamma, K^\times) = T(k)/R$ for $T = R^1_{K/k}(\mathbb{G}_m)$ [CTS1, Prop. 15]. The resulting isomorphism $\mathbf{SK}_1(A)(F) \cong T(k)/R$ is a specialisation morphism (in t_1 and t_2) [Wad, (6.9) & (6.10)] so that the composite $\mu_{n^2}(F) \to \mathbf{SK}_1(A)(F) \cong T(k)/R$ is the canonical morphism $\mu_{n^2}(k) \to T(k)/R$. It suffices to prove that the surjectivity of the latter.

First take a flabby resolution $1 \to S \to E \to T \to 1$ of K-split tori, then $H^1(k, S) = T(k)/R$ (loc. cit., Thm. 2). The evaluation morphism $S \times \hat{S} \to \mathbb{G}_m$ induces a perfect pairing [Nak, Tat]:

$$H^1(k, S) \times H^1(k, \hat{S}) \to H^2(k, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.$$

Moreover, $H^1(k, S) \cong H^1(\Gamma, S(K))$. This follows from the inflation-restriction exact sequence [GS, 3.3.14] and $H^1(K, S) = 0$. The pairing above can be modified to a pairing

$$H^1(\Gamma, S(K)) \times H^1(\Gamma, \hat{S}(K)) \to \mathrm{Br}(K/k) \cong \mathbb{Z}/n^2\mathbb{Z}.$$

Now note that $\mu_{n^2} \subset T$ so that we get a dual map $\hat{T} \to \mathbb{Z}/n^2\mathbb{Z}$. Using the flabby resolution and the pairing $T(k) \times \hat{T}(K) \to K^\times$, we get the following commutative diagram of pairings:

$$\begin{tikzcd}
H^1(k, S) & \times & H^1(k, \hat{S}) \arrow[d, "\cong"] \arrow[r] & H^2(k, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} \\
H^1(\Gamma, S(K)) \arrow[u, "\cong"] & \times & H^1(\Gamma, \hat{S}(K)) \arrow[d] \arrow[r] & \mathrm{Br}(K/k) \arrow[u, hook] \arrow[d, equal] \\
T(k) \arrow[u] & \times & H^2(\Gamma, \hat{T}(K)) \arrow[d] \arrow[r] & \mathrm{Br}(K/k) \arrow[d, equal] \\
\mu_{n^2}(k) \arrow[u] & \times & H^2(\Gamma, \mathbb{Z}/n^2) \arrow[r] & \mathrm{Br}(K/k).
\end{tikzcd}$$

The bottom pairing is perfect as $\mu_{n^2}(k) \cong \mathbb{Z}/n^2$; note that the bottom square comes from the compatibility of the pairings

$$\begin{tikzcd}
T(k) & \times & \hat{T}(K) \arrow[d] \arrow[r] & K^\times \arrow[d, equal] \\
\mu_{n^2}(k) \arrow[u] & \times & \mathbb{Z}/n^2 \arrow[r] & K^\times.
\end{tikzcd}$$

As $H^1(k,S) = T(k)/R \cong \mathbb{Z}/n$, to prove the surjectivity of $\mu_{n^2}(k) \to T(k)/R$, it suffices to prove the injectivity of $H^1(k,\hat{S}) \to H^2(\Gamma,\mathbb{Z}/n^2)$. Since $H^1(\Gamma,\hat{E}(K)) = 0$, this comes down to proving the injectivity of $H^2(\Gamma,\hat{T}) \to H^2(\Gamma,\mathbb{Z}/n^2)$. This morphism fits into an exact sequence

$$H^2(\Gamma, I_\Gamma) \to H^2(\Gamma,\hat{T}) \to H^2(\Gamma,\mathbb{Z}/n^2)$$

because of the exact sequence of group functors

$$0 \to \mu_{n^2} \to T \to R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m \to 0.$$

Clearly $T \to R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$ factors through $R_{K/k}(\mathbb{G}_m)$, so that $H^2(\Gamma, I_\Gamma) \to H^2(\Gamma,\hat{T})$ factors through $H^2(\Gamma,\mathbb{Z}[\Gamma])$ which is trivial by Shapiro's Lemma. This proves the desired injectivity. ■

Remark 3.25 – Note that the proof also defines an invariant of the torus T with values inside $\mathcal{H}^4_{n^2}$.

Conclusion

"Chi tace e chi piega la testa muore ogni volta che lo fa, chi parla e chi cammina a testa alta muore una volta sola."
— Giovanni Falcone

Overall, in this text we studied invariants of $\mathbf{SK}_1$. On the one hand, we defined wild invariants starting from existing moderate invariants using lifts and appropriate cycle modules. On the other hand, we compared invariants and proved ρ_{Kahn} is not trivial for Platonov's examples of non-trivial SK_1. This gives a different way of looking at Suslin's conjecture (Conjecture I.12).

Conjecture C.1

Let k a field and A a central simple k-algebra of $\mathrm{ind}_k(A)$ containing a square factor, then Suslin's invariant is not trivial for $\mathbf{SK}_1(A)$.

Remark C.2 – By Suslin's invariant, we mean either $\rho_{\mathrm{S06},A}$ or $\tilde{\rho}_{\mathrm{S06},A}$ depending on $\mathrm{char}(k)$ and $\mathrm{ind}_k(A)$. Clearly, a positive answer to this conjecture would imply Suslin's conjecture. Therefore, one could call this conjecture a *strong* version of Suslin's conjecture. For biquaternion algebras, this conjecture is true by Theorem I.20 and Remark 2.40. We can also rephrase this question for other invariants and obtain a modified conjecture.

Again, by the index reduction formula (Proposition I.14), it suffices to answer the question for central simple k-algebras A of $\mathrm{ind}_k(A) = p^2$ (p prime). Using Theorems I.16 and I.17, we can also reduce the question to verifying it for cyclic division algebras of the form $[(a,b)_p \otimes [(c,d)_p$ as in Proposition I.19.

We now try to attack this problem with the techniques from Chapters 2 and 3.

(a) Lifting and specialising invariants – By lifting central simple algebras from positive characteristic to characteristic zero as in §2.1.2 (a), we obtain the following result.

Proposition C.3

Let (K, R, k) be a p-triple, A a central simple k-algebra, and B the lifted R-Azumaya algebra. If Suslin's (strong) conjecture holds for A, then it also holds for B_K.

Proof. Recall that $\mathrm{ind}_k(A) = \mathrm{ind}_K(B_K)$. The statement on Suslin's conjecture follows from Corollary 2.3. The one on Suslin's strong conjecture holds as by definition ρ_{S06,B_K} maps to $\tilde{\rho}_{\mathrm{S06},A}$ under a morphism (see Theorem 2.43):

$$\mathrm{Inv}^4\left(\mathbf{SK}_1(B_K), \mathcal{H}^*_{r,L,\mathcal{B}_K}\right) \to \mathrm{Inv}^4\left(\mathbf{SK}_1(A), \mathcal{H}^*_{r,L,\mathcal{A}}\right).$$

■

Remark C.4 – Whether the inverse of Proposition C.3 holds is an open question and does not follow formally from the definition. Indeed, suppose $\mathbf{SK}_1(A) = 0$, i.e. $\mathrm{SK}_1(A \otimes_k k') = 0$ for any field extension k' of k. Then, $\mathrm{SK}_1(B_K \otimes_K K') = \mathrm{SK}_1(A \otimes_k k') = 0$ for any p-extension (K', R', k') of (K, R, k). But it is not sure that $\mathrm{SK}_1(B_K \otimes_K F) = 0$ for any extension F of K. If we reformulate this in the setting of §2.4.2; then the inverse translates into a possible injectivity of the morphism (2.13).

To the author, the constructions introduced in this thesis do not seem to give immediate ways of making strong reductions of characteristics. It would be however interesting to do so and to be able to define one of the dotted arrows in the "diagram" beneath where we abbreviate Suslin's conjecture to SC and Suslin's strong conjecture to SSC.

$$\begin{tikzcd}
\text{SC positive characteristic} \arrow[r, leftrightarrow, dotted, "?"] & \text{SC characteristic 0} \\
\text{SSC positive characteristic} \arrow[u] \arrow[r, leftrightarrow, dotted, "?"] & \text{SSC characteristic 0} \arrow[u]
\end{tikzcd}$$

(b) Comparing invariants – Using Theorem 3.24 and the Bloch-Kato isomorphism, we find the following result in moderate characteristic.

Corollary C.5

Let k be a field containing an l^2-th root of unity (for $l \neq \mathrm{char}(k)$ any prime) and let $A = (a,b)_l \otimes (c,d)_l$ be any product of two symbol k-algebras. If $\{a,b,c,d\} \neq 0 \in K_4^M(k)/l$, then $\mathbf{SK}_1(A) \neq 0$.

Proof. In characteristic 0, this follows immediately from the injectivity of φ (Remark 3.20) and $j(0,l) \not\equiv 0 \mod l$ (Theorem 3.24). In characteristic p, this follows analogously from (3.4). ■

By a result of Rost-Serre-Tignol, there is little hope that this gives a general way to approach Suslin's conjecture (in moderate characteristics). They prove that given k contains a primitive 4-th root of unity, the biquaternion k-algebra $(a,b) \otimes (c,d)$ is cyclic if and only if $\langle\langle a,b,c,d \rangle\rangle = 0 \in W_q(k)$ [RST, Thm. 3]. By Milnor's conjecture for quadratic forms (§3.1.2) the latter induces $\{a,b,c,d\} = 0 \in K_4^M(k)/2$. However Theorems I.16 and I.20 give cyclic biquaternion algebras A with $\mathbf{SK}_1(A) \neq 0$.

(c) Overall viewpoint – Apart from the questions posed above, it would also be interesting to find more examples of non-triviality of any of the existing invariants. It seems a very hard task to do so, but a small improvement could turn out to be a large step towards proving Suslin's conjecture.

Appendix A

Verification of cycle module rules

"Mathematics is no more computation than typing is literature."
— John Allen Paulos

In this appendix, we verify that $\mathcal{H}^*_{p^n,L}$ of Definition 2.23 verifies the rules of cycle modules as in §1.2 (a). Recall that the data D1-D4 are given in §2.2.1 (d), (e), and (f).

Proposition A.1

Let (K, R, k) be a p-triple with $(L, S, \overline{L})$ a finite Galois p-extension. Then, $\mathcal{H}^*_{p^n,L}$ of Definition 2.23 respects the rules R1a-R3e, FD, and C of cycle modules.

Proof. Rules R1a-R3e follow immediately from the definition of both $\mathcal{H}^*_{p^n,L}$ and its data D1-D4. Only rules R1c and R3b are maybe not straightforward obtainable. R1c relies on the universal property of tensor products. R3b is proved by passing to completions and using [Ser1, Ch. 2, Thm. 1] (see e.g. [GS, Cor. 7.3.11 & Prop. 7.4.1]). The proof of rule FD follows as in the classical case of finite support of divisors [Har, Ch. II. Lem. 6.1].

We deduce now rule C from the fact that it holds for Milnor K-groups [Kat4]. The residues ∂_K for Milnor K-groups are explained in §1.2 (d). To avoid a K-cacophony, we replace (K, R, k) by $(F, R, \overline{F})$. Let $\mathcal{X}$ be an integral R-scheme, local of dimension 2. We suppose first that the structure morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is surjective. Then, $X = \mathcal{X} \times_R F$ is an F-scheme and $Y = \mathcal{X} \times_R \overline{F}$ is an $\overline{F}$-scheme, both of dimension 1. Furthermore, $\mathrm{char}(F(X)) = 0$ and $\mathrm{char}(\overline{F}(Y)) = p$. So we have to verify that the composition of residues gives a complex (where y_0 is the closed point of $\mathcal{X}$ and $q \geq 2$):

$$H^{q+1}_{p^n,L}(F(X)) \to \bigoplus_{x\in X^{(1)}} H^q_{p^n,L}(F(x)) \oplus \bigoplus_{y\in Y^{(0)}} H^q_{p^n,L}(\overline{F}(y)) \to H^{q-1}_{p^n,L}(\overline{F}(y_0)). \tag{A.1}$$

We describe both the appearing groups and residues with K-groups as this allows us to use rule C for Milnor's K-groups. We start describing the groups by K-theory.

- *The group $H^{q+1}_{p^n,L}(F(X))$:*
  As
  $$\Gamma = \mathrm{Gal}(F_{\mathrm{nr}}(X)/F(X)) \cong \mathrm{Gal}(F_{\mathrm{nr}}/F) \cong \mathrm{Gal}(\overline{F}_s/\overline{F}),$$
  we know that $\mathrm{cd}_p(\Gamma) \leq 1$ [Ser2, Ch. II, Prop. 3]. The spectral sequence of Hochschild-Serre
  $$E_2^{s,t} = H^s\left(\Gamma, H^t(F_{\mathrm{nr}}(X), \mu_{p^n}^{\otimes q})\right) \Longrightarrow H^{s+t}(F(X), \mu_{p^n}^{\otimes q})$$
  induces an isomorphism
  $$H^1(\Gamma, H^q(F_{nr}(X), \mu_{p^n}^{\otimes q})) \cong \ker\big[H^{q+1}_{p^n}(F(X)) \to H^{q+1}_{p^n}(F_{\mathrm{nr}}(X))\big].$$
  Furthermore, the Bloch-Kato isomorphism gives us $H^q(F_{\mathrm{nr}}(X), \mu_{p^n}^{\otimes q}) \cong K_q(F_{\mathrm{nr}}(X))/p^n$. So, we get an isomorphism
  $$H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \cong \ker\big[H^{q+1}_{p^n}(F(X)) \to H^{q+1}_{p^n}(F_{\mathrm{nr}}(X))\big] \tag{A.2}$$
  and hence an inclusion
  $$H^{q+1}_{p^n,L}(F(X)) \subset H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n). \tag{A.3}$$

- *The group $H^q_{p^n,L}(F(x))$ for $x \in X^{(1)}$:*
  In the same way as above, we get an inclusion
  $$H^q_{p^n,L}(F(x)) \subset H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n). \tag{A.4}$$

- *The group $H^q_{p^n,L}(\overline{F}(y))$ for $y \in Y^{(0)}$:*
  Let $y \in Y^{(0)}$, then $H^q_{p^n}(\overline{F}(y)) \cong H^1\left(\overline{F}(y), \nu_n(q-1)_{\overline{F}(y)_s}\right)$ by (2.5). The isomorphism of Bloch-Kato-Gabber $\nu_n(q-1)_{\overline{F}(y)_s} \cong K_{q-1}(\overline{F}(y)_s)/p^n$ induces an isomorphism
  $$H^1(\overline{F}(y), K_{q-1}(\overline{F}(y)_s)/p^n) \cong H^{q+1}_{p^n}(\overline{F}(y)),$$

which also induces an inclusion:

$$
\begin{aligned}
&H^q_{p^n,L}(\overline{F}(y)) \\
&\quad\cong \ker\big[H^1\big(\overline{F}(y), K_{q-1}(\overline{F}(y)_s)/p^n\big) \to H^1\big(\overline{L}(y), K_{q-1}(\overline{F}(y)_s)/p^n\big)\big] \\
&\quad\subset \ker\big[H^1\big(\overline{F}(y), K_{q-1}(\overline{F}(y)_s)/p^n\big) \to H^1\big(\overline{F}_s(y), K_{q-1}(\overline{F}(y)_s)/p^n\big)\big].
\end{aligned}
\tag{A.5}
$$

This last term is isomorphic to $H^1\big(\Gamma, (K_{q-1}(\overline{F}(y)_s)/p^n)^{\Gamma_{\overline{F}_s(y)}}\big)$ by the inflation-restriction sequence [GS, Prop. 3.3.14].

- *The group $H^{q-1}_{p^n,L}(\overline{F}(y_0))$ for y_0 the closed point of $\mathcal{X}$:*
  As above:

  $$H^{q-1}_{p^n,L}(\overline{F}(y_0)) \subset H^1\left(\Gamma, (K_{q-2}(\overline{F}(y_0)_s)/p^n)^{\Gamma_{\overline{F}_s(y_0)}}\right). \tag{A.6}$$

Let us now explain the residues by means of K-theory.

- *The residue $\partial_x : H^{q+1}_{p^n,L}(F(X)) \to H^q_{p^n,L}(F(x))$ for $x \in X^{(1)}$:*
  The valuation attached to x induces a residue ∂_x, but also a Γ-equivariant residue $\partial_{K,x} : K_q(F_{\mathrm{nr}}(X))/p^n \to K_{q-1}(F_{\mathrm{nr}}(x))/p^n$ (as $\mathrm{Gal}(F_{\mathrm{nr}}(x)/F(x)) \cong \Gamma$). Hence this induces a morphism (which we give the same name by a slight abuse of notation):

  $$\partial_{K,x} : H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \to H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n).$$

  Lemma A.2 (infra) induces that $\partial_{K,x}$ is compatible with ∂_x under the inclusions (A.3) and (A.4) in a commutative diagram

  $$\begin{tikzcd}
  H^{q+1}_{p^n,L}(F(X)) \arrow[r, hook] \arrow[d, "\partial_x"'] & H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \arrow[d, "\partial_{K,x}"] \\
  H^q_{p^n,L}(F(x)) \arrow[r, hook] & H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n).
  \end{tikzcd} \tag{A.7}$$

- *The residue $\partial_y : H^{q+1}_{p^n,L}(F(X)) \to H^q_{p^n,L}(\overline{F}(y))$ for $y \in Y^{(0)}$:*
  Lemma A.2 shows that under the injection (A.5) $\mathrm{im}(\partial_y)$ ends up in

$H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n)$. On the other hand, the valuation attached to y induces a Γ-equivariant residue $\partial_{K,y} : K_q(F_{\mathrm{nr}}(X)) \to K_{q-1}(\overline{F}_s(y))$ and hence a morphism

$$\partial_{K,y} : H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \to H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n).$$

Lemma A.2 shows that we have a commutative diagram which explains the compatibility of ∂_y and $\partial_{K,y}$ under the inclusions (A.3) and (A.5):

$$\begin{tikzcd} H^{q+1}_{p^n,L}(F(X)) \arrow[r, hook] \arrow[d, "\partial_y"'] & H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \arrow[d, "\partial_{K,y}"] \\ H^q_{p^n,L}\left(\overline{F}(y)\right) \arrow[r, hook] & H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n). \end{tikzcd} \tag{A.8}$$

- *The residue $\partial^x_{y_0} : H^q_{p^n,L}(F(x)) \to H^{q-1}_{p^n,L}(F(y_0))$ for $x \in X^{(1)}$:*
  Lemma A.2 shows that under the inclusion (A.6) $\mathrm{im}(\partial^x_{y_0})$ is mapped into $H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n)$. On the other hand, we have a Γ-equivariant residue $\partial^x_{K,y_0} : K_{q-1}(F_{\mathrm{nr}}(x)) \to K_{q-2}(\overline{F}_s(y_0))$ giving on the cohomological level a morphism:

$$\partial^x_{K,y_0} : H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n) \to H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n).$$

  Again, Lemma A.2 guarantees that ∂^x_{K,y_0} is compatible with $\partial^x_{y_0}$ under the inclusions (A.4) and (A.6) so that we get a commutative diagram

$$\begin{tikzcd} H^q_{p^n,L}(F(x)) \arrow[r, hook] \arrow[d, "\partial^x_{y_0}"'] & H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n) \arrow[d, "\partial^x_{K,y_0}"] \\ H^{q-1}_{p^n,L}(\overline{F}(y_0)) \arrow[r, hook] & H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n). \end{tikzcd} \tag{A.9}$$

- *The residue $\partial^y_{y_0} : H^q_{p^n,L}(\overline{F}(y)) \to H^{q-1}_{p^n,L}(F(y_0))$ for $y \in Y^{(0)}$:*
  In this situation we also have a residue $\partial^y_{y_0}$ on the cohomology groups and a Γ-equivariant residue in K-theory $\partial^y_{K,y_0} : K_{q-1}(\overline{F}_s(y)) \to$

$K_{q-2}(\overline{F}_s(y_0))$ (for $y \in Y^{(0)}$). Then ∂^y_{K,y_0} induces a morphism on the cohomological level:

$$\partial^y_{K,y_0} : H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n) \to H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n).$$

Lemma A.2 shows once more a compatibility of ∂^y_{K,y_0} with $\partial^y_{y_0}$ under the inclusions (A.5) and (A.6):

$$\begin{tikzcd} H^q_{p^n,L}(\overline{F}(y)) \arrow[r, hook] \arrow[d, "\partial^y_{y_0}"'] & H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n) \arrow[d, "\partial^y_{K,y_0}"] \\ H^{q-1}_{p^n,L}(\overline{F}(y_0)) \arrow[r, hook] & H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n). \end{tikzcd} \tag{A.10}$$

In total, we have a collection of residues:

$$\begin{aligned} &H^1(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n) \longrightarrow \\ &\qquad \bigoplus_{x \in X^{(1)}} H^1(\Gamma, K_{q-1}(F_{\mathrm{nr}}(x))/p^n) \oplus \bigoplus_{y \in Y^{(0)}} H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n) \\ &\qquad\qquad \longrightarrow H^1(\Gamma, K_{q-2}(\overline{F}_s(y_0))/p^n). \end{aligned}$$

We know this is a complex as Milnor's K-groups respect rule C [Kat3]. The commutative diagrams (A.7,A.8,A.9,A.10) then show that (A.1) is a complex as well.

If the structure morphism is not surjective, $\mathcal{X}$ is either an F-scheme or an $\overline{F}$-scheme. If $\mathcal{X}$ is an F-scheme, the cycle module consists of kernels of usual (moderate) Galois cohomology groups. Rule C then follows immediately from rule C in the moderate case. If $\mathcal{X}$ is an $\overline{F}$-scheme, we can rewrite (A.1) using (2.5) and the isomorphism of Bloch-Kato-Gabber as

$$\begin{aligned} &H^1(\Gamma, K_q(\overline{F}_s(\mathcal{X}))/p^n) \to \bigoplus_{x \in \mathcal{X}^{(1)}} H^1(\Gamma, K_{q-1}(\overline{F}_s(x))/p^n) \\ &\qquad\qquad \to H^1(\Gamma, K_{q-2}(\overline{F}_s(x_0))/p^n), \end{aligned}$$

where x_0 is the closed point of $\mathcal{X}$. This is again a complex as the residues are again compatible with the residues from Milnor's K-theory (see Lemma A.2 in the case "y and y_0") and as rule C holds for Milnor's K-theory [Kat3]. ∎

Lemma A.2

Let $\mathcal{X}$ be an integral R-scheme, local of dimension 2 with surjective structure morphism, then the diagrams (A.7,A.8,A.9,A.10) are commutative.

Proof. We have to prove four situations, let us treat them case by case.

- *Diagram* (A.7) *is commutative for* $x \in X^{(1)}$*:*
  The Bloch-Kato isomorphism $K_q(F_{nr}(X))/p^n \cong H^q(F_{\mathrm{nr}}(X), \mu_{p^n}^{\otimes q})$ is defined by the Galois symbol and hence commutes with the usual residue on $H^q(F_{\mathrm{nr}}(X), \mu_{p^n}^{\otimes q})$ (with section given by the cup product with a class of an uniformiser π_x of the valuation associated with x) [GS, Prop. 7.5.1]. One deduces the result from this as the isomorphism (A.2) is an inflation and as ∂_x also has a section given by the cup product with the class of π_x.

- *Diagram* (A.8) *is commutative for* $y \in Y^{(0)}$*:*
  Recall that we also have to verify that $\mathrm{im}(\partial_y)$ is contained in $H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n)$. As the residue ∂_y is defined by a section, we can take $w \otimes \bar{x}_2 \otimes \ldots \otimes \bar{x}_q \in H^q_{p^n,L}(\overline{F}(y))$ with $w \in W_n(\overline{F}(y))$ and $x_2, \ldots, x_q \in \mathcal{O}_y^\times$ ($\mathcal{O}_y$ being the valuation ring corresponding to the valuation associated with y). If π_y is an uniformiser of $\mathcal{O}_y$, it is the residue of

  $$i(w) \cup h^q_{p^n,F(X)}(\{\pi_y, x_2, \ldots x_q\}) \in H^{q+1}_{p^n,L}(F(X)).$$

  Hence it corresponds to

  $$\left((\sigma(a) - a)\{\pi_y, x_2, \ldots, x_q\}\right)_\sigma \in H^1\left(\Gamma, K_q(F_{\mathrm{nr}}(X))/p^n\right),$$

  where $a^{(p)} - a = w$ with $a \in W_n(\overline{F}(y))$ and where we consider $(\sigma(a) - a)$ as an element of $\mathbb{Z}/p^n\mathbb{Z}$. On the other hand, $w \otimes \bar{x}_2 \otimes \ldots \otimes \bar{x}_q$ corresponds to

  $$\left((\sigma(a) - a)\{\bar{x}_2, \ldots, \bar{x}_q\}\right)_\sigma \in H^1(\Gamma, K_{q-1}(\overline{F}(y)_s)/p^n).$$

  This implies the commutativity and that $\left((\sigma(a) - a)\{\bar{x}_2, \ldots, \bar{x}_q\}\right)_\sigma$ is indeed an element of $H^1(\Gamma, K_{q-1}(\overline{F}_s(y))/p^n)$ as $\partial_{K,y}$ has its images in this group.

- *Diagram* (A.9) *is commutative for* $x \in X^{(1)}$*:*
  The verification follows in an analogous way as the previous case.

- *Diagram* (A.10) *is commutative for* $y \in Y^{(0)}$*:*
  The isomorphisms

  $$\nu_n(q-1)_{\overline{F}(y)_s} \cong K_{q-1}(\overline{F}(y)_s)/p^n, \qquad \nu_n(q-2)_{\overline{F}(y_0)_s} \cong K_{q-2}(\overline{F}(y_0)_s)/p^n,$$

  and the residue $K_{q-1}(\overline{F}(y)_s) \to K_{q-2}(\overline{F}(y_0)_s)$ induce a residue:

  $$\begin{array}{rcl} \nu_n(q-1)_{\overline{F}(y)_s} & \to & \nu_n(q-2)_{\overline{F}(y_0)_s}, \quad \text{defined by} \\ a \otimes \pi_0 \otimes x_2 \otimes \ldots \otimes x_{q-1} & \mapsto & \bar{a} \otimes \bar{x}_2 \ldots \otimes \bar{x}_{q-1}. \end{array}$$

  Here $a \in W_n(\mathcal{O}_v)$ and $x_i \in \mathcal{O}_v^\times$, where $\mathcal{O}_v$ is the valuation ring associated with the valuation v induced by y_0 with uniformiser π_0. By the definition of the residue $\partial^y_{y_0}$ (see Remarks 2.22 and 2.27), it is clear that these residues are compatible.

■

Appendix B

Elementary obstruction and Weil restriction

"The dream begins with a teacher who believes in you, who tugs and pushes and leads you to the next plateau, sometimes poking you with a sharp stick called 'truth'."
— Dan Rather

– Dedicated to the memory of Joost van Hamel –

In this appendix, we treat the subject of a first paper of the author [Wou1]. It is not related to questions about $\mathbf{SK}_1$, but rather concerns the existence of rational points on varieties. The methods used though are similar to the ones used in the main core of this article: Galois cohomology, homology, ... It is this setting that made the author familiar with these techniques. The authors owes a lot to Joost van Hamel for introducing him to this subject. This appendix is dedicated to his memory.

B.1 Introduction

For a field k and a variety X over k (i.e. a separated k-scheme of finite type), questions concerning k-rational points of X have been studied since ages. Different aspects arise in this area of research. In this appendix we focus on a certain obstruction to the existence of a rational point, namely the elementary obstruction, introduced by Colliot-Thélène and Sansuc [CTS2, Sec. 2.2].

In this appendix, we denote by $\overline{k}$ a separable closure[1] of k and Γ_k by Γ. If X is a smooth, geometrically integral variety over k, the elementary obstruction $\mathrm{ob}(X)$ of X is defined as the class of the exact sequence of left Γ-modules

$$\mathrm{OB}(X) = \quad 1 \to \overline{k}^{\times} \to \overline{k}(X)^{\times} \to \overline{k}(X)^{\times}/\overline{k}^{\times} \to 1$$

[1] This conflicts with the conventions posed for the rest of this thesis. This notation however keeps up with most of the publications on this subject.

as Yoneda extension in $\mathrm{Ext}^1_\Gamma(\overline{k}(X)^\times/\overline{k}^\times, \overline{k}^\times)$. Note that we use the common notation $\overline{k}(X)$ for the function field of $\overline{X} = X \times_k \overline{k}$. Analogously, we denote $\overline{k}[X]$ to be ring of regular functions on $\overline{X}$. If X contains a k-rational point, then $\mathrm{ob}(X) = 0$ [CTS2, Prop. 2.2.2]. Furthermore, if $\overline{k}[X]^\times = \overline{k}^\times$, the class of

$$E(X) = \quad 1 \to \overline{k}^\times \to \overline{k}(X)^\times \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 1$$

in $\mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X}), \overline{k}^\times)$ is denoted by $e(X)$. Colliot-Thélène and Sansuc show that the morphism

$$\delta : \mathrm{Ext}^1_\Gamma(\overline{k}(X)^\times/\overline{k}^\times, \overline{k}^\times) \to \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X}), \overline{k}^\times),$$

which arises in the long exact sequence induced by

$$1 \to \overline{k}(X)^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 1,$$

is injective and that $\delta(\mathrm{ob}(X)) = e(X)$ [CTS2, Prop. 2.2.4]. This is a consequence of Shapiro's Lemma and Hilbert 90. Therefore, it is also justified to say $e(X)$ is the elementary obstruction of X. In this paper we mainly use this definition for the elementary obstruction.

Several authors have been wondering whether the elementary obstruction behaves well under classical geometric constructions. A first observation is that the elementary obstruction is a birational invariant, since birationally equivalent varieties have isomorphic function fields. Wittenberg proves being zero behaves well under rational maps [Wit2, Lem. 3.1.2]. Borovoi, Colliot-Thélène, and Skorobogatov wonder whether being zero behaves well under base extension (i.e. whether $\mathrm{ob}(X) = 0$ implies $\mathrm{ob}(X \times_k K) = 0$ for K a field extension of k and X a smooth, geometrically integral variety over k) [BCTS, Sec. 2]. They give several (partial) positive answers to this question. Wittenberg gives a positive answer to this question for arbitrary (smooth, proper, geometrically integral) X when K is a p-adic or real closed field [Wit2, Cor. 3.2.3] or when k is a number field and the Tate-Shafarevich group of the Picard variety of X is finite [Wit2, Cor. 3.3.2]. He also gives a negative answer to this question by producing a counterexample over $\mathbb{C}((t))$ (unpublished).

In this appendix, we focus on the question whether being zero behaves well under the Weil restriction of varieties. To describe the problem more explicitly, we first recall the definition of the Weil restriction.

Definition B.1

Let k be a field and k' a finite field extension of k. Let X be a variety defined over k'. We say a variety $R_{k'/k}X$ over k is the Weil restriction (of scalars) of X to k if there is a k'-morphism $\varphi : R_{k'/k}X \times_k k' \to X$ such that for any k-variety Y and k'-morphism $f : Y \times_k k' \to X$, a unique k-morphism $g : Y \to R_{k'/k}X$ exists such that $\varphi \circ g' = f$. Here $g' : Y \times_k k' \to R_{k'/k}X \times_k k'$ is the k'-morphism induced by g. If the Weil restriction exists, it is unique up to k-isomorphism.

The following proposition guarantees the existence of the Weil restriction.

Proposition B.2

Let k be a field, $\overline{k}$ a separable closure, and k' a finite subextension of k in $\overline{k}$. Denote $\Gamma = \mathrm{Gal}(\overline{k}/k)$, $H = \mathrm{Gal}(\overline{k}/k')$, and let X be a quasiprojective variety over k'. The Weil restriction $R_{k'/k}X$ of X exists and

$$R_{k'/k}X \times_{k'} \overline{k} = \prod_{[\sigma] \in H\backslash\Gamma} \sigma X.$$

Here σX is the $\overline{k}$-variety obtained by base extension from $X \times_k \overline{k}$ by $\sigma : \overline{k} \to \overline{k}$ and $H\backslash\Gamma$ are the right cosets of H in Γ. The k'-morphism $\varphi : R_{k'/k}X \times_k k' \to X$ is obtained by descent theory from its base extension $\overline{\varphi} : \overline{R_{k'/k}X} \to \overline{X}$, the projection onto the factor $(\mathrm{id})X$.

For the proof, see [Mil2, Prop. 16.26]. Remark that if $[\sigma] = [\tau] \in H\backslash\Gamma$, the universal property of fibre products guarantees σX and τX to be isomorphic as $\overline{k}$-varieties. The universal property of the Weil restriction gives also a bijection between $R_{k'/k}X(k)$ and $X(k')$, as rational points are equivalent with sections of the structure morphism. It is then natural to ask the following question.

Question B.3

Let k be a field and k' a finite field extension. Suppose X is a smooth, geometrically integral variety over k' such that the Weil restriction $R_{k'/k}X$ exists. Does $e(X) = 0$ implies $e(R_{k'/k}X) = 0$ and vice versa?

We answer this question partially positively. First we give a result on product varieties, as the Weil restriction is closely related to product varieties by Proposition B.2.

B.2 Product varieties

Let X and Y be two smooth geometrically integral varieties over a field k, then the following theorem is a merely homological result.

Theorem B.4

The multiplication $\pi : \bar{k}(X)^\times/\bar{k}^\times \oplus \bar{k}(Y)^\times/\bar{k}^\times \to \bar{k}(X \times_k Y)^\times/\bar{k}^\times$ induces a morphism by pullback

$$\pi^{*\prime} : \mathrm{Ext}^1_\Gamma(\bar{k}(X \times_k Y)^\times/\bar{k}^\times, \bar{k}^\times) \to \mathrm{Ext}^1_\Gamma(\bar{k}(X)^\times/\bar{k}^\times, \bar{k}^\times) \oplus \mathrm{Ext}^1_\Gamma(\bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times)$$

such that $\pi^{*\prime}(\mathrm{ob}(X \times_k Y)) = (\mathrm{ob}(X), \mathrm{ob}(Y))$. If $\bar{k}[X]^\times = \bar{k}^\times = \bar{k}[Y]^\times$, then the Γ-morphism $\psi : \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times_{\bar{k}} \overline{Y})$, defined by pullback of linebundles, induces a morphism

$$\psi^{*\prime} : \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X} \times_{\bar{k}} \overline{Y}), \bar{k}^\times) \to \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X}), \bar{k}^\times) \oplus \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{Y}), \bar{k}^\times)$$

such that $\psi^{*\prime}(e(X \times_k Y)) = (e(X), e(Y))$. Even more, $\pi^{*\prime}$ and $\psi^{*\prime}$ commute with the natural inclusions:

$$\begin{tikzcd}
\mathrm{Ext}^1_\Gamma(\bar{k}(X \times_k Y)^\times/\bar{k}^\times, \bar{k}^\times) \arrow[r, "\pi^{*\prime}"] \arrow[d, "\delta"] & \mathrm{Ext}^1_\Gamma(\bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times) \oplus \mathrm{Ext}^1_\Gamma(\bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times) \arrow[d, "\delta"] \\
\mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X} \times_{\bar{k}} \overline{Y}), \bar{k}^\times) \arrow[r, "\psi^{*\prime}"] & \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X}), \bar{k}^\times) \oplus \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{Y}), \bar{k}^\times).
\end{tikzcd}$$

If π or ψ is an isomorphism, then $e(X \times_k Y) = 0$ (resp. $\mathrm{ob}(X \times_k Y) = 0$) if and only if $e(X) = 0$ and $e(Y) = 0$ (resp. $\mathrm{ob}(X) = 0$ and $\mathrm{ob}(Y) = 0$).

Remark B.5 – If X and Y are smooth geometrically integral varieties satisfying $\bar{k}[X]^\times = \bar{k}^\times = \bar{k}[Y]^\times$, then $X \times_k Y$ is also smooth geometrically integral and by a result of Rosenlicht [Ros1, Thm. 2] it satisfies $\bar{k}[X \times_k Y]^\times = \overline{k}^\times$. So speaking about $e(X \times_k Y)$ in the second case does make sense.

Proof. If we denote the canonical isomorphism

$$\mathrm{Ext}^1_\Gamma(\bar{k}(X)^\times/\bar{k}^\times \oplus \bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times) \to \mathrm{Ext}^1_\Gamma(\bar{k}(X)^\times/\bar{k}^\times, \bar{k}^\times) \oplus \mathrm{Ext}^1_\Gamma(\bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times)$$

by φ, then $\pi^{*\prime} = \varphi \circ \pi^*$ is the required morphism, where

$$\pi^* : \mathrm{Ext}^1_\Gamma(\overline{k}(X \times_k Y)^\times/\overline{k}^\times, \overline{k}^\times) \to \mathrm{Ext}^1_\Gamma(\bar{k}(X)^\times/\bar{k}^\times \oplus \bar{k}(Y)^\times/\bar{k}^\times, \bar{k}^\times)$$

is the pullback of 1-extensions by π. We now prove the assertion on the elementary obstruction.

We surely have a morphism of short exact sequences which consists of product morphisms:

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
\bar{k}^\times \oplus \bar{k}^\times \arrow[r, "\pi_1"] \arrow[d] & \bar{k}^\times \arrow[d] \\
\bar{k}(X)^\times \oplus \bar{k}(Y)^\times \arrow[r, "\pi_2"] \arrow[d] & \bar{k}(X \times_k Y)^\times \arrow[d] \\
\bar{k}(X)^\times/\bar{k}^\times \oplus \bar{k}(Y)^\times/\bar{k}^\times \arrow[r, "\pi_3=\pi"] \arrow[d] & \bar{k}(X \times_k Y)^\times/\bar{k}^\times \arrow[d] \\
1 & 1.
\end{tikzcd}$$

Denote the left short exact sequence by $E(X)\oplus E(Y)$. The right short exact sequence is $E(X\times_k Y)$. By the general theory of Yoneda extensions [ML, Ch. III], we get

$$\varphi^{-1}(e(X),e(Y)) = [\pi_1(E(X)\oplus E(Y))] = [E(X\times_k Y)\pi_3] = \pi^*(e(X\times_k Y)),$$

where $\pi_1(E(X)\oplus E(Y))$ denotes the pushforward of the Yoneda extension $E(X)\oplus E(Y)$ by π_1 and $E(X\times_k Y)\pi_3$ denotes the pullback of the Yoneda extension $E(X\times_k Y)$ by π_3. This proves the first part.

The second part is proved analogously using Γ-morphisms $\pi_4 : \mathrm{Div}(\overline{X})\oplus \mathrm{Div}(\overline{Y}) \to \mathrm{Div}(\overline{X}\times_{\bar{k}}\overline{Y})$ and $\psi : \mathrm{Pic}(\overline{X})\oplus\mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X}\times_{\bar{k}}\overline{Y})$. The commutativity assertion follows from the following morphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
\bar{k}(X)^\times/\bar{k}^\times \oplus \bar{k}(Y)^\times/\bar{k}^\times \arrow[r, "\pi_3"] \arrow[d] & \bar{k}(X\times_k Y)^\times/\bar{k}^\times \arrow[d] \\
\mathrm{Div}(\overline{X})\oplus\mathrm{Div}(\overline{Y}) \arrow[r, "\pi_4"] \arrow[d] & \mathrm{Div}(\overline{X}\times_{\bar{k}}\overline{Y}) \arrow[d] \\
\mathrm{Pic}(\overline{X})\oplus\mathrm{Pic}(\overline{Y}) \arrow[r, "\pi_5=\psi"] \arrow[d] & \mathrm{Pic}(\overline{X}\times_{\bar{k}}\overline{Y}) \arrow[d] \\
1 & 1.
\end{tikzcd}$$

This induces a morphism of long exact sequences, by Shapiro's lemma and Hilbert 90 containing the required diagram.

So we see that in any case $e(X)=0$ and $e(Y)=0$ (resp. $\mathrm{ob}(X)=0$ and $\mathrm{ob}(Y)=0$) if $e(X\times Y)=0$ (resp. $\mathrm{ob}(X\times Y)=0$). If ψ (resp. π) is an isomorphism, $\psi^{*\prime}$ (resp. $\pi^{*\prime}$) is so too, so in one of these cases the inverse implication holds as well (recall that $e(-)=0$ if and only if $\mathrm{ob}(-)=0$). ∎

Remark B.6 – A known result says that if $\overline{X}$ and $\overline{Y}$ are varieties over separable closed field $\overline{k}$, then as groups the morphism $\psi : \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})$, defined by pull-backs, has a section. This section restricts a line bundle on $\overline{X} \times_k \overline{Y}$ to $x_0 \times \overline{Y}$ and $\overline{X} \times y_0$ where x_0 and y_0 are base points on $\overline{X}$ and $\overline{Y}$. So as groups $\mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y})$ is a direct summand of $\mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})$. This looks interesting to get more information on the structure of $\mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y}), \overline{k}^\times)$.

In our case however, X and Y are defined over a not necessarily separably closed field k and $\psi : \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})$ is a Γ-morphism. The section however is not necessarily a Γ-morphism since the base points do not have to behave well (if we do not know anything about the existence of k-rational points on X and Y). So we cannot use this result to extend the previous theorem in a direct way. However, we do retrieve the injectivity of the Γ-morphism ψ.

Of course $\psi : \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times_{\bar{k}} \overline{Y})$ does not need to be an isomorphism, the product of an elliptic curve with itself delivering a counterexample [Har, Ch. IV, Ex. 4.10]. We can however give sufficient conditions for ψ to be an isomorphism. This involves the notion of the *relative Picard functor* and the *Picard variety*. If X is a smooth, geometrically integral, projective variety over a field k, we denote the relative Picard functor by $\mathcal{P}\mathrm{ic}_{X/k}$ (see definition in the proof of Proposition B.7), which is representable by a group variety $\mathbf{Pic}(X)$, the Picard variety. Denote by $\mathbf{Pic}^0(X)$ the zerocomponent of $\mathbf{Pic}(X)$. (See [BLR, Ch. 8] for more information.)

Proposition B.7

> If X is projective and $\mathbf{Pic}^0(\overline{X}) = 0$, then $\psi : \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})$ is a Γ-isomorphism.

Proof. By Remark B.6 we know that ψ is injective, so it is sufficient to prove $\operatorname{coker} \psi = 0$. By definition

$$\mathcal{P}\mathrm{ic}_{\overline{X}/\overline{k}}(\overline{Y}) = \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})/\mathrm{Pic}(\overline{Y}) \cong \mathrm{Hom}_{\overline{k}}(\overline{Y}, \mathbf{Pic}(\overline{X})).$$

Any $f \in \mathrm{Hom}_{\overline{k}}(\overline{Y}, \mathbf{Pic}(\overline{X}))$ has a connected image, but since $\mathbf{Pic}^0(\overline{X}) = 0$, the connected components of $\mathbf{Pic}(\overline{X})$ are its points. So $\mathrm{Hom}_{\overline{k}}(\overline{Y}, \mathbf{Pic}(\overline{X}))$

consists of the constant maps onto a point of $\mathbf{Pic}(\overline{X})$. This does not depend on Y, so

$$\mathrm{Hom}_{\overline{k}}(\overline{Y}, \mathbf{Pic}(\overline{X})) \cong \mathrm{Hom}_{\overline{k}}(\overline{k}, \mathbf{Pic}(\overline{X})) \cong \mathrm{Pic}(\overline{X}).$$

Because these isomorphisms are induced by the representability of the Picard functor,

$$\operatorname{coker} \psi = \frac{\mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})/\mathrm{Pic}(\overline{Y})}{\mathrm{Pic}(\overline{X})} \cong \frac{\mathrm{Pic}(\overline{X})}{\mathrm{Pic}(\overline{X})} = 0.$$

■

Proposition B.8

> If X is quasiprojective, $\mathrm{char}(k) = 0$, and $\mathrm{Pic}(\overline{X})$ is finitely generated, then $\mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \cong \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y})$.

Proof. Say $X \subset X_1$ for a projective variety X_1. Since $\mathrm{char}(k) = 0$, there exists a (smooth, projective) Hironaka desingularisation X' of X_1. As X is smooth, X is isomorphic to an open of X'. So without loss of generality we assume X to be an open part of X'. The exact sequence

$$\mathrm{Div}_{\overline{X'} \setminus \overline{X}}(\overline{X'}) \to \mathrm{Pic}(\overline{X'}) \to \mathrm{Pic}(\overline{X}) \to 0$$

induces $\mathrm{Pic}(\overline{X'})$ to be finitely generated, as $\mathrm{Pic}(\overline{X})$ and $\mathrm{Div}_{\overline{X'} \setminus \overline{X}}(\overline{X})$ are finitely generated. ($\mathrm{Div}_{\overline{X'} \setminus \overline{X}}(\overline{X})$ are the divisors on $\overline{X'}$ with support outside $\overline{X}$.)

It suffices to prove $\mathrm{Pic}(\overline{X'} \times_{\overline{k}} \overline{Y}) \cong \mathrm{Pic}(\overline{X'}) \oplus \mathrm{Pic}(\overline{Y})$ as this also induces $\mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y}) \cong \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y})$. Indeed, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(\overline{X'}) \oplus \mathrm{Pic}(\overline{Y}) \arrow[r] \arrow[d] & \mathrm{Pic}(\overline{X'} \times_{\overline{k}} \overline{Y}) \arrow[d] \\
0 \arrow[r] & \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(\overline{Y}) \arrow[r] \arrow[d] & \mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y}) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

where the vertical arrows are the surjective restriction morphisms. If the injection of the first row turns out to be an isomorphism, then the injection of the bottom row is also surjective, hence it is an isomorphism.

Because $\mathrm{Pic}(\overline{X'})$ is finitely generated, we have $\mathbf{Pic}^0(\overline{X'}) = 0$. Indeed, if $\mathbf{Pic}^0(\overline{X'}) \neq 0$, then $\mathbf{Pic}^0(\overline{X'})$ is an abelian variety of dimension $m > 0$ whose group of $\overline{k}$-points is finitely generated as $\mathrm{Pic}(\overline{X'}) = \mathrm{Hom}_{\overline{k}}(\overline{k}, \mathbf{Pic}(\overline{X'}))$ is finitely generated. On the other hand the group of $\overline{k}$-points of an abelian variety is divisible [Fre, Thm. 2]. But a divisible, non-trivial, finitely generated group does not exist. In this way we get a contradiction and so the proposition follows by Proposition B.7.

■

Consequently we obtain the following result.

Corollary B.9

Let X and Y be smooth, geometrically integral varieties over a field k with $\bar{k}[X]^\times = \bar{k}^\times = \bar{k}[Y]^\times$. Let $\overline{k}$ be a separable closure of k and $\Gamma = \mathrm{Gal}(\overline{k}/k)$. If one of the following conditions holds

(i) X is projective and $\mathbf{Pic}^0(\overline{X}) = 0$, or

(ii) X is quasiprojective, $\mathrm{char}(k) = 0$, and $\mathrm{Pic}(\overline{X})$ is finitely generated,

then

$$\psi^{*\prime} : \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X} \times_{\overline{k}} \overline{Y}), \overline{k}^\times) \to \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{X}), \overline{k}^\times) \oplus \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{Y}), \overline{k}^\times)$$

is an isomorphism such that $\psi^{*\prime}(e(X \times_k Y)) = (e(X), e(Y))$.

So if one of the conditions is true, $e(X \times_k Y) = 0$ if and only if $e(X) = 0$ and $e(Y) = 0$.

B.3 Weil restriction

Knowing more on the case of product varieties, we proceed to the Weil restriction. Throughout this section we assume that k' is a finite subextension of a field k in $\overline{k}$. Denote $H = \mathrm{Gal}(\overline{k}/k')$ and let X be a smooth, geometrically integral, quasiprojective variety over k'. The Weil restriction of X from k' to k exists by Proposition B.2 and we abbreviate it as $\mathcal{R}$.

Proposition B.10

The natural H-morphism $\overline{k}(X)^\times \to \overline{k}(\mathcal{R})^\times$ induces a pullback of 1-extensions

$$\Pi^* : \mathrm{Ext}^1_\Gamma(\overline{k}(\mathcal{R})^\times/\overline{k}^\times, \overline{k}^\times) \to \mathrm{Ext}^1_H(\overline{k}(X)^\times/\overline{k}^\times, \overline{k}^\times),$$

with $\Pi^*(\mathrm{ob}(\mathcal{R})) = \mathrm{ob}(X)$. If furthermore $\overline{k}[X]^\times = \overline{k}^\times$, then the natural H-morphism $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{\mathcal{R}})$ induces a pullback of 2-extensions

$$\Phi^* : \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \to \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{X}), \overline{k}^\times),$$

with $\Phi^*(e(\mathcal{R})) = e(X)$. As in Proposition B.4, these morphisms commute with the natural inclusions sending $\mathrm{ob}(-)$ to $e(-)$.

Remark B.11 – The natural H-morphisms mentioned in the proposition are induced by Proposition B.2. This proposition gives a k'-morphism $\varphi : \mathcal{R} \times_k k' \to X$ retrieved by descent from the $\overline{k}$-projection $\overline{\varphi} : \overline{\mathcal{R}} \to \overline{X}$. This morphism $\overline{\varphi}$ gives by pullback of principle divisors and line bundles the required H-morphisms.

Remark B.12 – As in Remark B.5 it is true that $\overline{k}[\mathcal{R}]^\times = \overline{k}^\times$ provided $\overline{k}[X]^\times = \overline{k}^\times$. So it makes sense to speak about $e(\mathcal{R})$ if at first glance we only require $\overline{k}[X]^\times = \overline{k}^\times$.

Proof. We give the proof of the assertion on 2-extensions. The assertion on 1-extensions follows in the same way. The commutative part follows as in Proposition B.4.

Denote the H-morphism $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{\mathcal{R}})$ by φ'. This induces a pullback

$$\varphi'^* : \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \to \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{X}), \overline{k}^\times).$$

If we use the forgetful map

$$\pi : \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \to \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times),$$

we get the required morphism $\Phi^* = \varphi'^* \circ \pi$. To prove $\Phi^*(e(\mathcal{R})) = e(X)$, we use the morphism $E(X) \to E(\mathcal{R})$ of H-extensions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{k}^{\times} & \longrightarrow & \overline{k}(X)^{\times} & \longrightarrow & \mathrm{Div}(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi'} & & \\
1 & \longrightarrow & \overline{k}^{\times} & \longrightarrow & \overline{k}(\mathcal{R})^{\times} & \longrightarrow & \mathrm{Div}(\overline{\mathcal{R}}) & \longrightarrow & \mathrm{Pic}(\overline{\mathcal{R}}) & \longrightarrow & 1.
\end{array}$$

As it is clear that the H-equivalence class of $E(\mathcal{R})$ equals $\pi([e(\mathcal{R})])$, we get from elementary homological reasons

$$\Phi^*(e(\mathcal{R})) = \varphi'^*(\pi([e(\mathcal{R})])) = [E(X)] = e(X).$$

■

So $e(\mathcal{R}) = 0$ implies $e(X) = 0$. We proceed figuring out when the converse is true. This holds in the very same situation as the converse holds for product varieties. To prove this, we use the notion of induced group module with some corresponding notation. Let G be a profinite group, H a subgroup of G, and A a left H-module, then the induced G-module is $\mathrm{Ind}_H^G(A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$ where $\mathbb{Z}[G]$ is considered as a right $\mathbb{Z}[H]$-module. This is a left G-module, the G-action is defined by $\gamma'(\gamma \otimes a) = \gamma'\gamma \otimes a$ for any $a \in A$ and $\gamma, \gamma' \in G$. If A and B are left H-modules and $f : A \to B$ is an H-morphism, then we get an induced G-morphism

$$\mathrm{Ind}_H^G(f) : \mathrm{Ind}_H^G(A) \mapsto \mathrm{Ind}_H^G(B), \quad \text{defined by} \quad \gamma \otimes a \mapsto \gamma \otimes f(a),$$

for $a \in A$ and $\gamma \in G$. If B is also a left G-module, we write $\mathrm{Ind}_H^G(f)'$ for the G-morphism $\pi \circ \mathrm{Ind}_H^G(f)$ with

$$\pi : \mathrm{Ind}_H^G(B) \to B \quad \text{defined by} \quad \gamma \otimes b \mapsto {}^{\gamma}b.$$

If E is an exact sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3,$$

then we get an induced exact sequence $\mathrm{Ind}_H^G(E)$:

$$\mathrm{Ind}_H^G(A_1) \xrightarrow{\tilde{f}_1} \mathrm{Ind}_H^G(A_2) \xrightarrow{\tilde{f}_2} \mathrm{Ind}_H^G(A_3),$$

where we have denoted $\tilde{f}_i = \mathrm{Ind}_H^G(f_i)$ for sake of simplicity.

Theorem B.13

If $\overline{k}[X]^\times = \overline{k}^\times$ and if one of the two following conditions is true

(i) X is projective and $\mathbf{Pic}^0(\overline{X}) = 0$, or

(ii) X is quasiprojective, $\mathrm{char}(k) = 0$, and $\mathrm{Pic}(\overline{X})$ is finitely generated,

then Φ^* of Proposition B.10 is an isomorphism.

Proof. We prove this result by giving another description of Φ^*.

If φ' is the H-morphism $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{\mathcal{R}})$ as defined in the proof of Proposition B.10, the induced Γ-morphism $\mathrm{Ind}_H^\Gamma(\varphi')' : \mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})) \to \mathrm{Pic}(\overline{\mathcal{R}})$ gives a pullback of 2-extensions:

$$\mathrm{Ind}_H^\Gamma(\varphi')'^* : \mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \to \mathrm{Ext}^2_\Gamma(\mathrm{Ind}_H^\Gamma \mathrm{Pic}(\overline{X}), \overline{k}^\times).$$

Furthermore say π' is the forgetful map

$$\pi' : \mathrm{Ext}^2_\Gamma(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times) \to \mathrm{Ext}^2_H(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times)$$

and let

$$i^* : \mathrm{Ext}^2_H(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times) \to \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{X}), \overline{k}^\times)$$

be the pullback by $i : \mathrm{Pic}(\overline{X}) \to \mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})) : \mathcal{L} \mapsto \mathrm{id} \otimes \mathcal{L}$. We have the following situation:

$$\begin{tikzcd}
\mathrm{Ext}^2_\Gamma(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \arrow[r, "\pi"] \arrow[d, "\mathrm{Ind}_H^\Gamma(\varphi')'^*"'] & \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{\mathcal{R}}), \overline{k}^\times) \arrow[r, "\varphi'^*"] & \mathrm{Ext}^2_H(\mathrm{Pic}(\overline{X}), \overline{k}^\times) \\
\mathrm{Ext}^2_\Gamma(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times) \arrow[rr, "\pi'"'] & & \mathrm{Ext}^2_H(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times). \arrow[u, "i^*"']
\end{tikzcd}$$

We prove $\Phi^* = \varphi'^* \circ \pi$ is an isomorphism by proving that $i^* \circ \pi' \circ \mathrm{Ind}_H^\Gamma(\varphi')'^*$ is an isomorphism and that the diagram above commutes. The latter follows directly from elementary homological reasons.

To prove the former, first observe that $i^* \circ \pi'$ is an isomorphism by Shapiro's Lemma as it has an inverse $\mathrm{Ind}_H^\Gamma(\mathrm{id})'_* \circ \mathrm{Ind}_H^\Gamma$ with

$$\mathrm{Ind}_H^\Gamma : \mathrm{Ext}_H^2(\mathrm{Pic}(\overline{X}), \overline{k}^\times) \to \mathrm{Ext}_\Gamma^2(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \mathrm{Ind}_H^\Gamma(\overline{k}^\times)) :$$
$$[E] \mapsto [\mathrm{Ind}_H^\Gamma(E)]$$

and $\mathrm{Ind}_H^\Gamma(\mathrm{id})'_*$ the pushforward

$$\mathrm{Ext}_\Gamma^2(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \mathrm{Ind}_H^\Gamma(\overline{k}^\times)) \to \mathrm{Ext}_\Gamma^2(\mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})), \overline{k}^\times)$$

by $\mathrm{Ind}_H^\Gamma(\mathrm{id})' : \mathrm{Ind}_H^\Gamma(\overline{k}^\times) \to \overline{k}^\times$. This is indeed an inverse by elementary homological reasons.

So it remains to prove $\mathrm{Ind}_H^\Gamma(\varphi')'^*$ is an isomorphism. We first choose a set of representatives $\{\sigma_1, \ldots, \sigma_n\}$ of the classes of $H\backslash\Gamma$ with $\sigma_1 = \mathrm{id}$.

If Condition (i) or (ii) is true, then pullback along all components

$$\psi : \bigoplus_{i=1}^n \mathrm{Pic}(\sigma_i X) \to \mathrm{Pic}(\overline{\mathcal{R}})$$

is an isomorphism of H-modules by Proposition B.7 and B.8. We prove there is a 1-1 correspondence $\tau : \mathrm{Ind}_H^\Gamma(\mathrm{Pic}(\overline{X})) \to \bigoplus_{i=1}^n \mathrm{Pic}(\sigma_i X)$ and that $\psi \circ \tau = \mathrm{Ind}_H^\Gamma(\varphi)'$. This induces $\mathrm{Ind}_H^\Gamma(\varphi)'$ to be an isomorphism.

First remark that for all $i = 1, \ldots, n$, base extension by σ_i induces a bijection $B_i : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\sigma_i X)$ which does not need to be a H-morphism as H does not necessarily commute with σ_i. There are also H-morphisms $\psi_i : \mathrm{Pic}(\sigma_i X) \to \mathrm{Pic}(\overline{\mathcal{R}})$ induced by projection on the i-th factor, so $\psi = \sum_{i=1}^n \psi_i$ and $\psi_1 = \varphi'$. It is easy to see that the B_i and ψ_i relate as ${}^{\sigma_i^{-1}}\psi_i(B_i(\mathcal{L})) = \psi_1(\mathcal{L})$ for any $\mathcal{L} \in \mathrm{Pic}(\overline{X})$.

To define τ, it satisfies defining $\tau(\gamma \otimes \mathcal{L})$ for any $\mathcal{L} \in \mathrm{Pic}(X)$ and $\gamma \in \Gamma$. Suppose $\gamma = \sigma_i h$ for $h \in H$ and $1 \leq i \leq n$, then we set $\tau(\gamma \otimes \mathcal{L})$ with 0 as $[\sigma_j]$-components for $j \neq i$ and $B_i({}^h\mathcal{L})$ as $[\sigma_i]$-component. This is well defined and as all the B_i are bijections, τ is indeed a 1-1 correspondence. Even more

$$\psi \circ \tau(\gamma \otimes \mathcal{L}) = \psi_i(B_i({}^h\mathcal{L})) = {}^{\sigma_i}\psi_1({}^h\mathcal{L}) = {}^\gamma\psi_1(\mathcal{L}) = \mathrm{Ind}_H^\Gamma(\varphi)'(\gamma \otimes \mathcal{L}).$$

■

So if one of the two conditions holds, $e(X) = 0$ if and only if $e(\mathcal{R}) = 0$.

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Glossary

In the glossary, k represents a field, F a field extension of it, A a central simple k-algebra, and X a k-scheme. For some definitions, we need further assumptions on the objects used. See the exact definition for the right assumptions.

$\langle a_1, \ldots, a_n \rangle$	quadratic n-form	66
$\langle\langle a_1, \ldots, a_n \rangle\rangle$	n-fold Pfister form	66
$\mathfrak{Ab}$	the category of commutative groups	ix
$[(a, b)$	either (a, b) or $[a, b)$	4
(a, b)	biquaternion k-algebra with $\mathrm{char}(k) \neq 2$	4
$[a, b)$	biquaternion k-algebra with $\mathrm{char}(k) = 2$	4
$[a, b)_p$	p-algebra	4
$(a, b)_p$	symbol algebra	4
$[(a, b)_p$	either $(a, b)_p$ or $[a, b)_p$	4
$A^i(X, M_j)$	i-th homology group of weight j of the Gersten complex associated with X and M	21
$A^i(X, M_j)_{\mathrm{mult}}$	multiplicative subgroup of $A^0(X, M_j)$	23
A_F	base extension of A to F	ix
$\tilde{A}^0(X, M_j)$	reduced subgroup of $A^0(X, M_j)$	23
$\sim_{\mathrm{Br}}$	Brauer-equivalent	2
$\mathrm{Br}(F/k)$	$\ker[\mathrm{Br}(k) \to \mathrm{Br}(F)]$	2
$\mathrm{Br}(k)$	Brauer group of k	2
${}_n\mathrm{Br}(k)$	part of n-torsion of $\mathrm{Br}(k)$	14
$\mathrm{cd}(k)$	cohomological dimension of k	ix
$\mathrm{cd}_p(k)$	p-cohomological dimension of k (for a prime p)	ix
$\mathrm{char}(k)$	characteristic of k	
$C^q_{p^n}(k)$	logarithmic differentials of k ($\mathrm{char}(k) = p$)	42
$\deg(A)$	degree of A	3
$D^q_{p^n}(k)$	$W_n(k) \otimes (k^\times)^{\otimes q}$ ($\mathrm{char}(k) = p$)	41

F_{nr}	maximal unramified extension of a discrete valued field F	x
$(F, \mathcal{O}_v, \kappa(v))$	valuation triple associated with a discrete valuation v on F	36
$\mathrm{Gal}(F/k)$	Galois group of F over k	
Γ_K	absolute Galois group of k	ix
$\mathbb{G}_m$	$\mathrm{Spec}(\mathbb{Z}[T, T^{-1}])$	ix
$\mathfrak{Groups}$	the category of groups	ix
$H_m^{i+1}(F)$	$H_{p^l}^{i+1}(F) \oplus H_r^{i+1}(F)$ if $\mathrm{char}(F) = p$ and $m = p^l r$ with $p \nmid r$	14, 41
$H_{n,A^{\otimes r}}^{i+1}(F)$	relatif $H_n^{i+1}(F)$ with respect to $A^{\otimes r}$	16
$\mathcal{H}_{m,L}^*$	cycle module associated with $H_m^{i+1}(F)$	21, 45
$\mathcal{H}_{n,L,\mathcal{A}^{\otimes r}}^*$	relatif cycle module associated with $H_{n,A^{\otimes r}}^{i+1}(F)$	21, 48, 57
$H_{p^n,\mathrm{nr}}^{i+1}(F)$	unramified cohomology	43
$h_{p,F}^n$	differential symbol of F of degree n ($\mathrm{char}(F) = p$)	46
$h_{m,F}^n$	Galois symbol of F of degree $n \in F^\times$ and weight m	15
$I(k)$	fundamental ideal of $W(k)$	67
$\mathrm{ind}_k(A)$	index of A	3
$\mathrm{Inv}^j(\mathbf{G}, M)$	invariants of degree j of a group functor $\mathbf{G}$ in a cycle module M	23
$I^n W_q(k)$	$I^n(k) \cdot W_q(k)$	67
$\overline{I^n W_q(k)}$	$I^n W_q(k) / I^{n+1} W_q(k)$	67
$I^n W_q'(k)$	$I^n(k) \cdot W_q'(k)$	67
$\overline{I^n W_q'(k)}$	$I^n W_q'(k) / I^{n+1} W_q'(k)$	67
$J_q(k)$	certain subgroup of $D_{p^n}(k)$ ($\mathrm{char}(k) = p$)	42
$\kappa(v)$	residue field of a discrete valuation v	x
$\overline{k}$	algebraic closure of k	ix
k-$\mathfrak{fields}$	the category of field extensions of k	ix
$(K/k, \sigma, a)$	cyclic algebra	3
$K_n(F)$	n-th Milnor K-group of F	15
k_s	separable closure of k	ix

$k((t_1))\dots((t_n))$	n-fold iterated Laurent series field over k	ix
$M_n(k)$	matrix algebra of $n\times n$ matrices over k	
μ_m	the Γ_k-module of m-th roots of unity in k_s	ix
$\mu_m(k)$	m-th roots of unity in k	x
$\bar{n}$	integer defined using a prime decomposition of n	27
$N_{F/k}$	norm of a finite field extension F of k	5
$\mathrm{Nrd}_{A/k}$	reduced norm of A	5
$\mathrm{Nrp}_{\sigma/k}$	Pfaffian norm of A	65
$\nu_n(q)$	kernel of the Cartier morphism	42
Ω^q_k	q-differentials on k	41
$\mathcal{O}_v$	valuation ring of a discrete valuation v	x
$\mathrm{per}_k(A)$	period of A	3
$\mathbf{PGL}_\infty$	projective linear group scheme	32
$\mathbf{PGSp}(A,\sigma)$	certain group scheme associated with A with symplectic involution σ	69
$\mathbf{Pic}(X)$	Picard variety of X	107
$\mathcal{P}\mathrm{ic}_{X/k}$	Picard functor of X	107
$\mathrm{Prd}_{a/k}(X)$	reduced characteristic polynompial of $a\in A$	5
$\mathrm{Prp}_{\sigma,a/k}(X)$	Pfaffian characteristic polynompial of $a\in A$	65
R-$\mathfrak{fields}$	the category of R-algebras which fields	18
$\rho_{\mathrm{BI},A}$	KMRT's invariant of $\mathbf{SK}_1(A)$ with A a biquaternion k-algebra	67
$\rho_{\mathrm{Kahn},A}$	Kahn's 2006 invariant of $\mathbf{SK}_1(A)$	27
$\tilde\rho_{\mathrm{Kahn},A}$	Kahn's 2006 generalised invariant of $\mathbf{SK}_1(A)$	58
$\rho_{r,A}$	Kahn's r-th invariant of $\mathbf{SK}_1(A)$	27
$\tilde\rho_{r,A}$	Kahn's r-th generalised invariant of $\mathbf{SK}_1(A)$	58
$\rho_{\mathrm{Rost},A}$	Rost's invariant of $\mathbf{SK}_1(A)$ with A a biquaternion k-algebra	25
$\rho_{\mathrm{S06},A}$	Suslin's 2006 invariant of $\mathbf{SK}_1(A)$	26
$\tilde\rho_{\mathrm{S06},A}$	Suslin's 2006 generalised invariant of $\mathbf{SK}_1(A)$	58
$\rho_{\mathrm{S91},A}$	Suslin's 1991 invariant of $\mathbf{SK}_1(A)$	25
$\tilde\rho_{\mathrm{S91},A}$	Suslin's 1991 generalised invariant of $\mathbf{SK}_1(A)$	58
$R^1_{k'/k}(\mathbb{G}_m)$	$\ker(R_{k'/k}(\mathbb{G}_m)\to\mathbb{G}_m)$	85
$R_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m$	$\mathrm{coker}(\mathbb{G}_m\to R_{k'/k}(\mathbb{G}_m))$	85

$R_{k'/k}(Y)$	Weil restriction of scalars to k	103
$\mathrm{SB}(A)$	Severi-Brauer variety of A	26
$\mathfrak{Sets}$	the category of sets	ix
$\mathrm{SK}_1(A)$	reduced Whitehead group of A	5
$\mathbf{SK}_1(A)$	reduced Whitehead group functor of A	8
$\mathrm{SL}_1(A)$	k-points of the special linear group of A	9
$\mathbf{SL}_1(A)$	special linear group of A	9
$\mathrm{Symd}(A, \sigma)$	symmetrised elements in A under involution σ	65
$\hat{T}$	dual of a torus T	85
$\mathrm{Tr}_{F/k}$	trace of a finite field extension F of k	5
$\mathrm{Trd}_{A/k}$	reduced trace of A	5
$\mathrm{Trp}_{\sigma/k}$	Pfaffian trace of A	65
$W(k)$	Witt ring of k	65
$W_n(k)$	Witt p-vectors of length n on k ($\mathrm{char}(k) = p$)	41
$W_q(k)$	Witt group of k	65
$W'_q(k)$	subgroup of $W_q(k)$ consisting of even-dimensional non-singular quadratic spaces	67
$X^{(i)}$	set of points of codimension i of X	x
$X(F)$	F-rational points of X	ix
X_F	base extension of X to F	ix

Index

Azumaya algebra, 31

biquaternion algebra, 11
Bloch-Kato isomorphism, 15
Bloch-Kato-Gabber isomorphism, 46
Brauer class, 2
Brauer group, 2
Brauer's decomposition theorem, 9
Brauer-equivalence, 2

cohomological invariant, 13
cyclic algebra, 3

differential symbol, 46
discrete valuation, x
division algebras, 1
dual torus, 85

elementary obstruction, 101
equicharacteristic, x

flasque resolution, 86
flasque torus, 86
fundamental ideal, 67

Galois symbol, 15
Gersten complex, 21
group functor, 13

Hensel's lemma à la Grothendieck, 32
hyperbolic involution, 67

index reduction formula, 9
invariant, 13
involution, 64

KMRT's invariant, 63
Kneser-Tits problem, 9

lifted Azumaya algebra, 31
logarithmic differentials, 41

Milnor K-groups, 15
Milnor's conjectures, 68
mixed characteristic, x
Morita invariance of SK_1, 6
Morita isomorphism, 5
multiplicative subgroup, 23

orthogonal involution, 64

p-algebra, 4
p-extension, 36
p-ring, 33
p-triple, 36
Pfaffian characteristic polynomial, 65
Pfaffian norm, 65
Pfaffian trace, 65
Pfister form, 66
Picard functor, 107
Picard variety, 107
Platonov's examples, 7

R-field, 18
R-valuation triple, 36
reduced Whitehead group, 6
residue morphism, 16
rigidity, 32

Severi-Brauer variety, 26
special linear group, 6
splitting field, 2
Suslin's conjecture, 8
Suslin's strong conjecture, 89
symbol algebra, 4
symmetrised elements, 65

symplectic involution, 64

Tannaka-Artin problem, 7
Tate twist, 14

unramified cohomology, 43
unramified division algebra, 32

valuation triple, 36

Wang's theorem, 7
Wedderburn's theorem, 1
Weil restriction, 102
Weil restriction of scalars, 103
Whitehead group, 5
Witt group, 65
Witt ring, 65
Witt vectors, 34
Witt-equivalence, 65

And now, the end is near,
And so I face the final curtain.
My friends, I'll say it clear,
I'll state my case of which I'm certain.

I've lived a life that's full,
I've travelled each and every highway.
And more, much more than this,
I did it my way.

Frank Sinatra

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